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Charles Clement Adderley,

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Oxford.

THE
ELEMENTS
OF
NATURAL OR EXPERIMENTAL
PHILOSOPHY.

BY
TIBERIUS CAVALLO, F.R.S. &c.

ILLUSTRATED WITH COPPER PLATES.

IN FOUR VOLUMES.

VOL. I.

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THE
P R E F A C E.

THE principal characteristic which distinguishes the human being from the rest of the animal creation, is the inheritance of knowledge, which the individuals of each generation are able to derive from their predecessors.

The bee of modern times forms the cells of its hive exactly of the same shape as the bee of the remotest antiquity; each species of birds builds its nest after the same unalterable pattern, and sings the same invariable melody. The sheep of the present day has no better defence against the wolf; nor has the fly against the spider, nor the smaller birds against the eagle, than the like animals of former times. The same wants, similar dangers, the like defects, and unalterable customs, are the constant attendants of each different tribe; nor is any

A 2 individual

individual benefited by the experience or by the improvements of all its predecessors.

Man alone has received from his Divine Creator the inestimable advantage of being benefited by the knowledge of his forefathers, and of his being able to bequeath that knowledge, together with his own improvements, to his posterity.

The accumulated experience of a long series of years, accurately recorded in a vast many books, or traditionally imparted from one generation to the other, gradually exalts the state of human beings, supplies their wants, increases their security, and promotes their happiness. The plough, the loom, the forge, the press, the glass-house, and innumerable other useful inventions of our predecessors, successively improved by constant use and experience, form the invaluable advantages of modern times; and their combined effect, actually elevates the individuals of a modern civilized nation, so far above the uninstructed savages, as might almost seem to render them of a different species.

That experience, properly disposed under distinct heads, forms the various subjects of knowledge. The arrangement, and the elucidation of each particular subject, is called a Science. The ultimate or the practical application of it is called an Art.

Arts and sciences are too numerous and too extended, to be comprehended in their greatest extent by each single individual: hence is derived the division of labour, or the adoption of a particular branch by each single individual. But all those branches derive their origin from the same natural powers, they are all in their principles regulated by the same general laws of Nature, and almost all their applications may be subjected to calculation and demonstration. The investigation of their origin, and of their mutual dependence on each other, the illustration of their principles, the methods of enlarging their limits by means of experiments and calculation, and their application to our various wants, fall under the title of NATURAL or EXPERIMENTAL PHILOSOPHY, the ELEMENTS of which form the subject of this Work.

In the course of the last twenty or thirty centuries, during which time (as written documents inform us) more or less attentive observations have been made on the properties of natural bodies, various theories have been formed, or different ideas have been entertained concerning the nature of those bodies, or concerning the general subject of Natural Philosophy; but the small proportion of real facts, and the vastly greater proportion of vague and unwarrantable ideas which formed those theories, rendered them always insufficient, and frequently absurd; whence confusion of ideas, and retardation of science, naturally ensued.

The nature and the fate of those theories gradually cautioned the judicious part of the inquisitive world, and shewed them the necessity of substituting experiments and strict mathematical reasoning to the suggestions of the imagination. This rational reform, or cautious mode of proceeding, since the 16th century, has been productive of a vast number of useful discoveries; and, by its having placed the progress of science in the right channel,

channel, has enabled philosophers to trace out the principles of several of its branches, to investigate divers new subjects, and to open new paths to the inexhaustible treasures of nature.

The progress of experimental investigation, and the mathematical mode of reasoning, are both slow and laborious; but they are safe, and productive of true and useful knowledge: nor has the human being any other means of feeling his way through the dark labyrinth of Nature. It is wonderful to observe what manual labour, and what exalted exertions of the human mind, have been bestowed upon the various branches of Natural Philosophy. Those profound inquiries, sometimes fruitless, and at other times either directly or indirectly successful, alternately display the strength and the weakness of the human understanding; but upon the whole, it must be acknowledged that wonderful improvements have undoubtedly been derived from those extraordinary exertions; and the progress of science within the last two centuries has certainly advanced with increasing velocity.

It is not my intention to deceive the reader by asserting, that I have rendered all the principles of Natural Philosophy intelligible to the meanest capacity ; for in that case, I should either have been obliged to omit the more abstruse branches of philosophy, or the fallacy of the assertion would be rendered staringly manifest in several of the following pages. Original discoveries of facts, or principles or laws of nature, are generally made through intricate and perplexed paths. By subsequent revision and consideration, the superfluous is removed, the defective is supplied, and the confused materials are properly arranged ; whence the train of reasoning frequently becomes shorter and more natural, or the nature of the subject is rendered more evident and more intelligible. But this simplification has a limit which differs in different subjects ; nor can the comprehension of what depends upon a vast number of previous ideas, mathematically connected, be rendered attainable to such persons as are destitute of such ideas, or whose mind is incapable of retaining the necessary chain of reasoning.

By

By following the example of the clearest writers, and by considering each particular subject in different points of view, I have endeavoured to explain it with all the simplicity and the clearness which my slender ability could suggest. In several places I have avoided some abstruse technical formalities of order or phraseology, and have preferred familiar expressions wherever it appeared practicable; but when the subject seemed less likely to be comprehended by the greatest number of readers, I have always placed it in the Notes, where those only who are competently qualified may read it. And here it must be observed, that, for the sake of distinction, the references from the text to those notes, consist of the common numerical figures; whereas the references to other notes containing quotations, additional remarks, &c. consist of asterisks, or such like marks.

A few repetitions, which the reader will meet with in the course of the work, will, I trust, be easily excused, considering that they have been thought necessary for promoting the elucidation of particular subjects. With
respect

respect to the termination of certain words of an entire Latin origin, it must be observed that I have indiscriminately written them either with a Latin or with an English termination, such as *radii* and *radiuses*, *media* and *meidums*, &c. for having found them used both ways by different writers, I was unwilling to adopt a decided partiality for either mode.

With respect to the disposition of the materials throughout the work, it may perhaps be necessary to mention, that my rule has been to begin with the general properties of matter, or such as constant experience shews to belong to bodies of every kind. I have afterwards proceeded to examine those which belong to a particular set of bodies, and then those of fewer or of single bodies.

The astronomical part has been naturally placed after the statement of the above-mentioned properties, since the knowledge of the appearances of the celestial bodies is not so immediately concerned with our welfare, as that of the substances which nearly

*

surround

surround us, and of which our very bodies are formed.

In the illustration of the various branches, a multiplicity of experiments and extended historical accounts have been carefully avoided, lest the statement of superfluities should have occupied the place of useful materials. The different subjects of Natural Philosophy cannot be rendered sufficiently intelligible without a certain extent of explanation; but at the same time their number would render the work too extensive, if the limits of absolute necessity were not carefully preserved. In this, however, the Author is exposed to a dangerous dilemma, as the same illustration which proves prolix to certain readers, is insufficient for others. Different views of the same abstruse subject, though tedious to the proficient, are undoubtedly of great assistance to the novice. In this case the limits of sufficiency or of insufficiency are vague and indeterminate; and whilst they tend to perplex the author, they afford, according to the inclination of the reader, ample scope for criticism or satisfaction. Natural order, accuracy

curacy of statements, perspicuity, and conciseness, have been the constant objects of my views in the compilation of this work. I have endeavoured to select from multiplicity, and to remove obscurity. In certain places I have added new facts, in others I have searched for new and true explanations of natural effects. I have pointed out the defects of several particulars, and have recommended the elucidation of the same to the diligence of zealous students. But whether or not the performance is sufficiently conformable to those views, I humbly submit it to the decision of the impartial and distinguishing part of my readers.

As this work is likely to fall into various hands, it may perhaps be useful to add a few remarks and a few directions for the use, not of the proficient, but of those to whom the subject is either partially or entirely new, in order that unprofitable labour, or extravagant expectations, may in great measure be avoided.

Of the various readers of books in general, I shall briefly attempt to discriminate the following

lowing classes. There are some who imagine that the same velocity of reading is sufficient for a novel, or a poetical, or an historical, or a scientific book; and when they find that they are not able to comprehend the latter, they conclude either that the author is obscure, or that they themselves have not capacity sufficient for it. Others imagine that a single careful perusal of a scientific book is sufficient to instruct them in a new subject. Lastly, there are others who never proceed to the next page, unless they have thoroughly understood the preceding part of the work. This method, in the reading of natural philosophy, though very proper, is by no means very pleasing, and generally tires the student before he has read a quarter of the work.

Where a great many new ideas must be acquired, much attention must be necessarily bestowed. Therefore, in the reading of novels or poetry, the only exertion of the mind which is required for a satisfactory perusal, is the connection of the different parts or accounts, and a tolerable degree of attention to the beauties of the performance
which

which arise from the style and the imagination of the writer ; for with respect to facts and the meaning of words, they are so much like the occurrences of common life, as never to demand any exertion of the understanding. Nearly the same thing may be said with respect to the reading of history. But with scientific subjects the case is quite different ; for in them the great variety of new things and new ideas, to which words of uncommon use have been appropriated, and their dependance upon each other, or upon facts of unusual occurrence, demand a continual exertion both of the memory and of the understanding ; which, unless it be relieved by means of order, patience, and a competent allowance of time, will certainly prove irksome to most students.

Therefore on those accounts I beg leave to recommend the following method. Let the novice in the study of natural philosophy read this work a first time, rather slowly, but without perusing those notes which, as has been remarked above, have a numerical reference, nor caring, as he proceeds, whether he

he does or does not thoroughly understand or remember all the particulars. By this means he will acquire a superficial idea of the subject, the meaning of several words will become familiar to him, and he will in all probability be delighted by the various and wonderful properties of the material world. He may then commence a second perusal, proceeding rather slower, endeavouring thoroughly to understand every part, and every figure of the plates, and examining, according to his ability, a greater or a lesser part of the notes. During this second perusal, it would be highly useful to attend a course of experimental lectures; for by this means the various objects, machines, actions, &c. will be easily and permanently fixed in his mind.

During this second reading he should pay particular attention to the numerous technical words, the meaning of which, whenever he forgets them, may be easily found out by recurring to the Index at the end of the work, which has for this purpose been rendered

much

much more copious than is customary for books of this kind. Lastly, the student may read over a third time, or oftener, such parts only of the work as his particular inclination, or his understanding or his memory may render necessary.

T. C.

WELLS STREET,
JAN. 1st, 1803.

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of the First Volume.*

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*must be placed at the end
of the Second Volume.*

The next EIGHT, viz. as far as Plate XXV. inclusively,
*must be placed at the end
of the Third Volume.*

And the last FOUR Plates ; viz. XXVI. to XXIX.
inclusively - - *must be placed at the end
of the Fourth Volume.*

The ASTRONOMICAL TABLE (a Quarter of a Sheet of
Letter Press) to face - p. 190 *of the Fourth Volume.*

ELEMENTS OF NATURAL PHILOSOPHY.

CHAPTER I.

Of NATURAL PHILOSOPHY;—its Name;—its Object;—its Axioms;—and the Rules of Philosophizing.

THE word Philosophy; though used by ancient authors in senses somewhat different, does, however, in its most usual acceptation, mean *the love of general knowledge*. It is divided into *moral* and *natural*. Moral philosophy treats of the manners, the duties, and the conduct of man, considered as a rational and social being; but the business of natural philosophy, is to collect the history of the phenomena which take place amongst natural things, viz. amongst the bodies of the Universe; to investigate their causes and effects; and thence to deduce such natural laws, as may afterwards be applied to a variety of useful purposes*.

Natural

* The word *philosophy* is of Greek origin. Pitagoras, a learned Greek, seems to have been the first who called himself philosopher; viz. a lover of knowledge, or of wisdom,

Natural things means all bodies; and the assemblage or system of them all is called the *universe*.

The word *phenomenon* signifies an appearance, or, in a more enlarged acceptation, whatever is perceived by our senses*. Thus the fall of a stone, the evaporation of water, the solution of salt in water, a flash of lightning, and so on; are all phenomena.

As all phenomena depend on *properties* peculiar to different bodies; for it is a property of a stone to fall towards the earth, of the water to be evaporable, of the salt to be soluble in water, &c. therefore we say that the business of natural philosophy is to examine the properties of the various bodies of the universe, to investigate their causes, and thence to infer useful deductions.

Agreeably

dom, from the words φίλος, a *lover* or *friend*, and σοφίας, of *knowledge* or *wisdom*. Moral philosophy is derived from the latin *mos*, or its plural *mores*, signifying manners or behaviour. It has been likewise called *ethics*, from the Greek ἠθες, *mos*, *manner*, *behaviour*. Natural philosophy has also been called *physics*, *physiology*, and *experimental philosophy*. The first of those names is derived from φύσις, *nature*, or φυσικός, *natural*; the second is derived from φύσις, *nature*, and λόγος, a *discourse*; the last denomination, which was introduced not many years ago, is obviously derived from the just method of experimental investigation, which has been universally adopted since the revival of learning in Europe.

* *Phenomenon*, whose plural is *phenomena*, owes its origin to the Greek word φαίνω, *to appear*.

Agreeably to this, the reader will find in the course of this work, an account of the principal properties of natural bodies, arranged under distinct heads, with an explanation of their effects, and of the causes on which they depend, as far as has been ascertained by means of reasoning and experience; he will be informed of the principal hypotheses that have been offered for the explanation of facts, whose causes have not yet been demonstratively proved; he will find a statement of the laws of nature, or of such rules as have been deduced from the concurrence of similar facts; and, lastly, he will be instructed in the management of philosophical instruments, and in the mode of performing the experiments that may be thought necessary either for the illustration of what has been already ascertained, or for the farther investigation of the properties of natural bodies.

We need not say much with respect to the end or design of natural philosophy.—Its application and its uses, or the advantages which mankind may derive therefrom, will be easily suggested by a very superficial examination of whatever takes place about us. The properties of the air we breathe; the action and power of our limbs; the light, the sound, and other perceptions of our senses; the actions of the engines that are used in husbandry, navigation, &c.; the vicissitudes of the seasons, the movements of the celestial bodies, and so forth; do all fall under the consideration of

4 Of PHILOSOPHY in general;

the philosopher. Our welfare, our very existence, depends upon them.

A very slight acquaintance with the political state of the world, will be sufficient to shew, that the cultivation of the various branches of natural philosophy has actually placed the Europeans and their colonies above the rest of mankind. Their discoveries and improvements in astronomy, optics, navigation, chemistry, magnetism, mineralogy, and in the numerous arts which depend on those and other branches of philosophy, have supplied them with innumerable articles of use and luxury, have multiplied their riches, and have extended their powers to a degree even beyond the expectations of our predecessors.

1
2 The various properties of matter may be divided into two classes, viz. the *general properties*, which belong to all bodies, and the *peculiar properties*, or those which belong to certain bodies only, exclusively of others.

In the first part of this work we shall examine the general properties of matter. Those which belong to certain bodies only, will be treated of in the second. In the third part we shall examine the properties of such substances as may be called *hypothetical*; their existence having not yet been satisfactorily proved. In the fourth we shall extend our views beyond the limits of our Earth, and shall examine the number, the movements, and other properties of the celestial bodies.

The

The fifth, or last part, will contain several detached articles, such as the description of several additional experiments, machines, &c. which cannot conveniently be inserted in the preceding divisions.

The axioms of philosophy, or the axioms which have been deduced from common and constant experience, are so evident and so generally known, that it will be sufficient to mention a few of them only.

I. Nothing has no property; hence,

II. No substance, or nothing, can be produced from nothing.

III. Matter cannot be annihilated, or reduced to nothing.

Some persons may perhaps not readily admit the propriety of this axiom; seeing that a great many things appear to be utterly destroyed by the action of fire; also that water may be caused to disappear by means of evaporation, and so forth. But it must be observed, that in those cases the substances are not annihilated; but they are only dispersed, or removed from one place to another, or they are divided into particles so minute as to elude our senses. Thus when a piece of wood is placed upon the fire, the greatest part of it disappears, and a few ashes only remain, the weight and bulk of which does not amount to the hundredth part of that of the original piece of wood. Now in this case the piece of wood is divided into

its component substances, which the action of the fire drives different ways: the fluid part, for instance, becomes steam, the light coaly part either adheres to the chimney or is dispersed through the air, &c. And if, after the combustion, the scattered materials were collected together, (which may in great measure be done), the sum of their weights would equal the weight of the original piece of wood.

IV. Every effect has, or is produced by, a cause, and is proportionate to it.

It may in general be observed with respect to those axioms, that we only mean to assert what has been constantly shewn, and confirmed by experience, and is not contradicted either by reason, or by any experiment. But we do not mean to assert that they are as evident as the axioms of geometry; nor do we in the least presume to prescribe limits to the agency of the Almighty Creator of every thing, whose power and whose ends are too far removed from the reach of our understandings.

Having stated the principal axioms of philosophy, it is in the next place necessary to mention the rules of philosophizing, which have been formed after mature consideration, for the purpose of preventing errors as much as possible, and in order to lead the student of nature along the shortest and safest way, to the attainment of true and useful knowledge.—Those rules are not more than four; viz.

I. We

I. We are to admit no more causes of natural things, than such as are both true and sufficient to explain the appearances.

II. Therefore to the same natural effects we must, as far as possible, assign the same causes.

III. Such qualities of bodies as are not capable of increase or decrease, and which are found to belong to all bodies within the reach of our experiments, are to be esteemed the universal qualities of all bodies whatsoever.

IV. In experimental philosophy we are to look upon propositions collected by general induction from phenomena, as accurately or very nearly true, notwithstanding any contrary hypotheses that may be imagined, till such time as other phenomena occur, by which they either may be corrected, or may be shewn to be liable to exceptions. X

With respect to the degree of evidence which ought to be expected in natural philosophy, it is necessary to remark, that physical matters cannot in general be capable of such absolute certainty as the branches of mathematics.—The propositions of the latter science are clearly deduced from a set of axioms so very simple and evident, as to convey perfect conviction to the mind; nor can any of them be denied without a manifest absurdity. But in natural philosophy we can only say, that because some particular effects have been constantly produced under certain circumstances; therefore they will most likely continue to be produced as long

as the same circumstances exist; and likewise that they do, in all probability, depend upon those circumstances. And this is what we mean by *laws of nature*; as will be more particularly defined in the next chapter.

We may, indeed, assume various physical principles, and by reasoning upon them, we may strictly demonstrate the deduction of certain consequences. But as the demonstration goes no farther than to prove that such consequences must necessarily follow the principles which have been assumed, the consequences themselves can have no greater degree of certainty than the principles are possessed of; so that they are true, or false, or probable, according as the principles upon which they depend are true, or false, or probable. It has been found, for instance, that a magnet, when left at liberty, does always direct itself to certain parts of the world; upon which property the mariner's compass has been constructed; and it has been likewise observed, that this directive property of a natural or artificial magnet, is not obstructed by the interposition or proximity of gold, or silver, or glass, or, in short, of any other substance, as far as has been tried, excepting iron and ferrugineous bodies. Now assuming this observation as a principle, it naturally follows, that, iron excepted, the box of the mariner's compass may be made of any substance that may be most agreeable to the workman, or that may best answer other purposes. Yet it must be
 confessed,

confessed, that this proposition is by no means so certain as a geometrical one; and strictly speaking it may only be said to be highly probable; for though all the bodies that have been tried with this view, iron excepted, have been found not to affect the directive property of the magnet or magnetic needle; yet we are not certain that a body, or some combination of bodies, may not hereafter be discovered, which may obstruct that property.

Notwithstanding this observation, I am far from meaning to encourage scepticism; my only object being to shew that just and proper degree of conviction which ought to be annexed to physical knowledge; so that the student of this science may become neither a blind believer, nor a useless sceptic*.

Besides a strict adherence to the abovementioned rules, whoever wishes to make any proficiency in the study of nature, should make himself acquainted with the various branches of mathematics; at least with the elements of geometry, arithmetic, trigonometry, and the principal properties of the conic

* *Scepticism* or *skepticism* is the doctrine of the *sceptics*, an ancient set of philosophers, whose peculiar tenet was, that all things are uncertain and incomprehensible; and that the mind is never to assent to any thing, but to remain in an absolute state of hesitation and indifference. — The word *sceptic* is derived from the Greek *σκηπτικός*, which signifies *considerate*, and *inquisitive*.

conic sections; for since almost every physical effect depends upon motion, magnitude, and figure, it is impossible to calculate velocities, powers, weights, times, &c, without a competent degree of mathematical knowledge; which science may in truth be called the language of nature.

CHAPTER II.

CONTAINING A GENERAL IDEA OF MATTER,
AND ITS PROPERTIES.

*Of the ELEMENTS; — and the Definitions of Words
that are principally used in Natural Philosophy.*

THE matter or substance of the bodies which we see, feel, taste, or, in short, that affect our senses, becomes known to us merely by its properties. We know that the sun exists because we see its luminous and circular shape; because we feel its heat. We know that the ground exists because we see it, and feel it with our limbs. We acknowledge the existence of air, because we feel the resistance it offers to the motion of other bodies, &c. Now the sun, the ground, the air, and all other bodies, must, agreeably to the first axiom, consist of something. That something is called *matter*; yet we are perfectly ignorant of the intimate nature of that matter; since we are unable to say whether

whether it consists of particles of any definite size, shape, and hardness; and whether all the bodies of the universe consist of the same sort of matter differently modified, or of different sorts of matter; and in the latter case, we can form no conjecture respecting the possible number of those different sorts.

Some philosophers have steadily supported that there is one sort only of original matter, and that the variety of bodies arises from the various arrangements of that primitive matter; like paste, bread, and biscuits, which may be said to consist of the same matter, viz. flour.

Other philosophers have believed that the sorts of primitive matter, or *elements*, are two. Others, that they are three. Others again, that they are four or five, or six or seven, and so on. But the history of suppositions must not be mistaken for the knowledge of facts*.

The truth is, that the present state of knowledge does not furnish us with reasons sufficient to determine

* Whoever wishes to examine the various ideas that have been entertained concerning original matter, and the number of the elements, must consult the works of the following authors; but his labour will not be compensated by any material information: Aristotle; Plato; Boyle, on the principles of natural bodies; Newton's Optics; Woodward's Nat. Hist. of the Earth, p. v.; Musschenbroek's Elements of Phys. § 61, 83, 383; Keill Introd. to Nat. Phil. Lect. viii.; Higgins on Light; Chambers's Cyclop; and Hutton's Mathem. Dict. Art. *Element*.

mine the number of the elements. Modern chemists, indeed, find from the result of their numerous analyses, that all the bodies which have been subjected to experiments, are either the substances that are mentioned in the following list, or they are a combination of some of them. Yet no great confidence should be placed upon their number; for besides there being great suspicion that several of them are still resolvable into simpler components; new substances are almost daily discovered by the present rapid progress of philosophical investigation; and some of them are merely hypothetical.

The following list contains the bare names of those elementary substances which are at present acknowledged by the philosophical chemists, or such as chemists have not yet been able to decompose; but a full explanation of the same will be found in other chapters of this work; and till then the reader is requested not to endeavour to investigate the meaning of their names, or to take any farther notice of them.

Light,	Phosphorus,
Calorific, or Caloric,	Radical muriatic,
The Electric Fluid,	Radical boracic,
The Magnetic Fluid,	Radical fluoric,
Oxygen,	Radical succinic,
Hydrogen,	Radical acetic,
Azote,	Radical tartaric,
Carbon,	Radical pyro-tartaric,
Sulphur,	Radical oxalic,
	Radical

Radical gallic,	Nickel,
Radical cytric,	Cobalt,
Radical malic,	Bismuth,
Radical benzoic,	Antimony,
Radical pyro-lignic,	Zinc,
Radical pyro-mucic,	Iron,
Radical camphoric,	Tin,
Radical lactic,	Lead,
Radical sach-lactic,	Copper,
Radical formic,	Mercury,
Radical prussic,	Silver,
Radical sebacic,	Platina,
Radical bombic,	Gold,
Radical laccic,	Silica,
Radical fuberic,	Argill,
Radical zoonic,	Baryt,
Arsenic,	Strontian,
Molybdenite,	Lime,
Tungsten,	Magnesia,
Chrome,	Jargonia,
Titanite,	Vegetable alkali,
Sylvanite,	Fossil alkali, and
Uranite,	Volatile alkali.
Manganese,	

Though most of the words that frequently occur in the subject of the present work, are generally used in common language, yet the accuracy of philosophical descriptions suggests the necessity of defining their meanings with a greater degree of precision, in order to avoid, as much as possible,
any

any ambiguity of sense, or any uncertainty of expression.

Besides those which are mentioned in the present chapter, there are several other words which deserve likewise to be particularly defined; but those we shall explain occasionally in the course of the work, and when the mind of the reader may be better disposed to comprehend their meanings.

Space (though it be incapable of a proper definition) may be said to be that universal and unlimited expanse in which all bodies are contained; and that part of space, which is occupied by any particular body, is called the *place of that body*.

Space is distinguished into *absolute*, and *relative*. *Absolute space* is that which is referred to nothing, and remains always similar and immoveable. *Relative space* is the same with absolute space in magnitude and figure, but not in situation. Suppose, for example, that a ship stood perfectly immoveable in the universe, the space which is contained within its cavity, would be called *absolute space*. But if the ship be in motion, then the same space within it will be called *relative space*.

Place is likewise distinguished into *absolute* and *relative*; the former being immoveable and permanent; whereas the latter refers to other bodies. Thus if a man be seated in a corner of a ship whilst the ship is sailing along, he is said to remain in the same place *relatively* to the parts of the ship; yet he is continually changing his *absolute place*.

Rest is the permanence of any body in the same place, and it is called *absolute rest*, or *relative rest*, as the place, which the body occupies, is either absolute or relative.

Motion, on the contrary, is a continual and successive change of place. And it is called *absolute motion* or *relative motion*, according as the change of situation is made in absolute or in relative place.

Thus, if a ship were to remain immoveable in the universe, a man sitting in a corner of it, would be said to be absolutely at rest; but if the ship be in motion whilst the man remains sitting, then this man will be said to be at rest relatively to the parts of the ship, though he is actually or absolutely in motion.—Farther, suppose that the ship were to move equably forward over a distance equal to its length, and that at the same time the man in his chair were drawn from the fore to the back part of the ship, with the same equable motion, then the man would be in motion relatively to the parts of the ship; yet he would remain in the same absolute place.

With respect to the words *matter* and *body*, we shall for the present only remark the following difference between them; viz. that the word *matter* has no relation to any determinate figure; whereas the word *body* more generally means some separate and determinate quantity of matter. Thus we say with propriety, that *the movements of the celestial bodies are difficultly determined, and the matter which*
forms

forms the atmosphere is heterogeneous; whereas it would be improper to change the places of the words *body* and *matter*, by saying that *the movements of the celestial matters are difficultly determined, and the body which forms the atmosphere is heterogeneous.*

Time, strictly speaking, is incapable of definition, and the only thing we can remark with respect to it, is the difference between *absolute* and *relative time*. *Absolute time* flows equably, but does not refer to the motion of bodies. *Relative time* is that portion of absolute time, during which a certain movement is performed, and we assume some of those movements, when they are equably and steadily performed, as the measures of time. Thus that portion of absolute time which the sun employs in performing its apparent revolution round the earth, is called *a day*; the 24th part of that day is called *an hour*; 365 times that day is called *a year*, and so on.

The *properties* of a thing are those *qualities* and *operations* which belong to that thing, and by which it is distinguished from other things that do not possess the same properties. It is, for instance, a property of the sun to be luminous, of the magnet to attract iron, &c.

The *hardness* of a body is that degree of resistance which the body offers to any power that may be applied for the purpose of separating its parts. Whereas *fluidity* is the want of that resistance; so that a perfect fluid is that body whose parts may
be

be separated by the application of the least force. It will appear from the sequel that we are not acquainted with any perfectly hard, or any perfectly fluid, body; so that we can only examine the intermediate gradations, which exist between those extremes; but those gradations which are expressed by the words *hardness, rigidity, brittleness, toughness, softness, clamminess, fluidity, &c.* are incapable of precise definitions or limits.

Cause and *effect* are relative terms; the *effect* being that which is produced by the cause, and the *cause* that which produces the effect.

Causes as well as effects are distinguished into *primary, secondary, &c.* or into *immediate* and *remote*. Thus when the heat of the sun rarefies the air, that rarefaction produces wind, and that wind impels a ship forward. In this case the heat of the sun is the cause of the wind; the wind is the effect of the rarefaction, and is at the same time the cause of the ship's motion; the motion of the ship is effected by the action of the wind, so that the wind is the immediate, and the heat of the sun is the remote, cause of the ship's motion.

A law of nature, or mechanical law, is a general effect, which has been constantly observed to take place under certain determinate circumstances*. Thus we know from constant and universal experience, that whenever a body is left to itself, it al-
ways

* *Mechanical* means something that relates to, or is regulated by, the nature and laws of motion.

ways falls towards the centre of the earth, unless some other body prevents it; we therefore assume this observation as a law of nature, and express it by saying, *that the various bodies of the earth tend, or gravitate, towards the centre of it.*

The existence or non-existence of a *vacuum*, meaning an extension entirely void of matter, has often been disputed amongst philosophers; their arguments always depending upon some assumed hypothesis concerning the intimate nature of matter or of its ultimate particles; but as we are utterly ignorant of the nature and properties of those particles, their arguments cannot determine the question one way or the other.—The only conclusions we can make with respect to a *vacuum*, are 1st. that the possibility of its existence can be easily imagined; 2dly. that we are not certain whether it really exist or not; and lastly, that if it be admitted that the figure of the least particles of matter is unchangeable, the motions of bodies, such as continually take place in the universe, cannot be understood without admitting the existence of a vacuum.

The word *infinity* has likewise been productive of numerous disputes. Many odd positions have been assumed for the support of specious arguments, and several absurd consequences have been deduced from them. Those errors have principally arisen from the idea of something determinate, which has been annexed to the words *infinite*, or *infinity*, instead

stead of something indefinite or indeterminate. In consequence of this idea, infinities have often been compared together, and one infinite has been said to be the double, or treble, or the half, &c. of another infinite; whereas infinities, (in the true sense of the word, which means something greater or less than any assignable quantity, but not determinate) are incapable of comparison; since an indeterminate quantity cannot bear any assignable proportion to another indeterminate quantity; and of course one infinite cannot be said to be greater than, equal to, or less than, another infinite.

It has been usually alledged, that if a line be infinitely extended one way only, and another line be infinitely extended both ways; the latter infinite line must be double the former infinite line, which evidently implies a limited or determinate length; namely, that the latter line has been extended on either side as much as the former line has been extended one way only.

Again; take the length of one inch, and suppose it to be divided into an infinite number of parts. Take also the length of a foot, and suppose this to be divided into an infinite number of parts. Here, they say, it is evident that the latter infinite is exactly equal to twelve times the former. But this, in my humble opinion, seems to be a mistaken conclusion; for the expressions of infinity do not refer to the extensions of one foot and one inch; but to the numbers of the parts into which those exten-

sions have been divided; and those numbers can bear no assignable proportion to each other; just because they are infinite.

The fact then is, that one foot is equal to twelve times one inch; and if each of those extensions be divided into any number of parts equal to each other in length, the number of parts in the extension of one foot will be equal to twelve times the number of the parts that are contained in the extension of one inch; but this is not the meaning of dividing a foot or an inch into an infinite number of parts; therefore when the foot and the inch, are each divided into an infinite number of parts, those numbers have no assignable proportion to each other; though the *sum* of the former is undoubtedly equal to twelve times the sum of the latter*.

* Numerous instances of an infinite number of quantities having a finite or determinate sum, occur both in arithmetic and in geometry. In geometry it is shewn, that a finite line may be divided into an infinite number of parts; and it is evident that the sum of all those parts must be equal to the line itself; viz. a finite quantity. In arithmetic it is shewn, besides many other instances, that if you take one half, and one half of that half, and one half of the last half, and so on without end, the sum of them all is equal to one; that is $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64}$, &c. = 1.

CHAPTER III.

OF THE GENERAL, OR COMMON, PROPERTIES
OF MATTER.

IT has been already remarked, that a body is distinguished from other bodies by means of its peculiar properties. Thus we know water by its fluidity, and by its want of taste, smell, and colour; gold is known by its great weight and peculiar colour; salt is known by its particular taste; and so forth. But there are certain properties, which belong equally to water, to gold, to salt, and to all other bodies. Extension for instance is a property which belongs to them all; for they all are extended. So likewise is weight; for they all are more or less heavy. Such then are called *General, or Common, Properties of Matter*; and, as far as we know, they are six in number; viz. *extension, divisibility, impenetrability, mobility, vis inertia, or passiveness, and gravitation.*

We have said above, as far as we know, because matter in general may possess other properties, that are not yet come to our knowledge. And the same observation may be made with respect to the universality of those properties; viz. that they are said to be general, because no body was ever found wanting any one of them. But mankind is not acquainted with all the bodies of the universe, and

even several of those which are known to exist, cannot be subjected to experiments.

Extension of a body is the quantity of space which a body occupies; the extremities of which, limit or circumscribe the matter of that body. It is otherwise called the *magnitude*, or *size*, or *bulk* of a body.

A certain quantity of matter may indeed be very small, or so fine as to penetrate the pores of most other bodies; but yet some extension it must have; and it is by the comparison of this property that one body is said to be larger than, equal to, or smaller than, another body. The measurement of a body consists in the comparison of the extension of that body with a certain determinate extension, which is assumed for the standard, such as an inch, a foot, a yard, a mile; and hence we say that a certain body is three feet long, another body is the hundredth part of an inch in length, and so on*.

A body is not only extended, but it is extended three different ways, viz. it has *length*, *breadth*, and *thickness*. Thus an ordinary sheet of writing paper is about sixteen inches long, fourteen inches broad, and

* The attempts, which have been made for the purpose of establishing an invariable standard of measure, together with the difficulties which obstruct the perfect accomplishment of that object; as also the principal measures which are now in use, will be mentioned in another part of this work.

and one hundredth part of an inch thick. Either of those dimensions might be called the length, or the breadth, or the thickness; but, by general custom, the longest extension is called the length, the next is called the breadth, and the shortest is called the thickness.

The outside of a body; the boundary, or that which lies contiguous to other bodies that are in contact with it, is called *the surface* of that body, and it has two dimensions only, viz. length and breadth; but it cannot have any thickness, for if it had thickness, it would not be the outside of the body; yet a surface by itself cannot exist. We indeed talk of surfaces independent of matter, as when we compare one surface with another, or describe the methods of measuring and dividing surfaces. In those cases, however, the surfaces exist in our imagination only, and even then our ideas have a reference to body. In short, our senses cannot perceive a surface without the existence of a body; or, more properly speaking, the outside of a body cannot exist without the body itself.

As the surface is the outside or boundary of a body, so a *line* is the boundary of a surface; suppose, for instance, that a surface is divided into two parts, the common boundary of the two parts is called a *line*, which has one extension, viz. length only.

A *point* is the beginning, or the end, of a line, and of course it has no extension; it being defined

by the mathematicians, that which has no parts nor magnitude. Thus if you divide a line into two parts, the division or boundary between the two parts is *a point*.

Having shewn above that there cannot exist, or that our senses cannot perceive, a surface without a body, it evidently follows that neither a line nor a point can be perceived without a body. We speak of the line or path of a planet; we also say that a stone thrown horizontally describes a curve line; but in those cases the meaning is, that the planet, or the stone, has passed through certain places; not that those lines do actually exist as any thing substantial. When we look on a sheet of paper, we see its surface, the edge of which is a line, and the extremity of the line, or corner, of the paper, is a point. But if you remove the paper; the surface, the line, and the point, vanish from our sight, and they can only remain in our imagination.

Divisibility of matter is the property of its being divisible into parts. Some philosophical writers have considered it as a distinct property of matter itself; but it may with more propriety be considered as a property of extension; for we can easily conceive that a given extension may be divided into any number of parts, let it be ever so great; but it is by no means known whether matter is, or is not, capable of being divided *ad infinitum*, that is, without any limit.

That a certain extension, as an inch, or any other length, be it ever so small, is capable of in-

finite division, may be rendered evident by means of arithmetic or of geometry. We may take, for instance, the halves of the proposed extension, then the halves of those parts, then the halves of those halves, and so on without end; for if you proceed in this manner ever so far, there will after all still remain the halves of the last parts, which may be also divided into other halves, &c. Again, suppose the line AB in fig. 1st. plate 1. to be the proposed extension. Through the extreme points of this line draw two indefinite lines EF, and CD, parallel to each other. In one of those lines, as EF, take a point L, and from this point draw straight lines to any parts of the line BD, every one of which lines will evidently cut the proposed extension AB into a different point. Now as the line BD may be produced towards D without limitation, and straight lines may be drawn from L to an infinite number of points in the extended line BD; therefore the extension AB may be divided without end, or beyond any assignable number of parts.

Thus far we have shewn that extension may be divisible into an unlimited number of parts; but with respect to the limits of the divisibility of matter itself we are perfectly in the dark. We can indeed divide certain bodies into surprisngly fine and numerous particles, and the works of nature offer many fluids and solids of wonderful tenuity; but both our efforts, and those naturally incul objects,

objects, advance a very short way towards infinity. Ignorant of the intimate nature of matter, we cannot assert whether it may be capable of infinite division, or whether it ultimately consists of particles of a certain size, and of perfect hardness.

I shall now add some instances of the wonderful tenuity of certain bodies, that has been produced either by art, or that has been discovered by means of microscopical observations amongst the stupendous works of nature.

The spinning of wool, silk, cotton, and such like substances, affords no bad specimens of this sort; since the thread which has been produced by this means, has often been so very fine as almost to exceed the bounds of credibility, had it not been sufficiently well authenticated. Mr. Boyle mentions, that two grains and a half of silk was spun into a thread 300 yards long.

A few years ago a lady of Lincolnshire spun a single pound of woollen-yarn into a thread 168000 yards long, which is equal to 95 English miles*. Also a single pound weight of fine cotton-yarn was lately spun, in the neighbourhood of Manchester, into a thread 134400 yards long.

The ductility of gold likewise furnishes a striking example of the great tenuity of matter amongst the productions of human ingenuity. A single grain weight of gold has been often extended into
a surface

* This lady's name at that time was Miss Ives. It is now Mrs. Ayre.

a surface equal to 50 square inches. If every square inch of it be divided into square particles of the hundredth part of an inch, which will be plainly visible to the naked eye, the number of those particles in one inch square will be 10000; and, multiplying this number by the 50 inches, the product is 500000; that is, the grain of gold may be actually divided into at least half a million of particles, each of which is perfectly apparent to the naked eye. Yet if one of those particles be viewed in a good microscope, it will appear like a large surface, the ten-thousandth part of which might by this means be easily discerned.

An ingenious artist in London has been able to draw parallel lines upon a glass plate, as also upon silver, so near one another, that 10000 of them occupy the space of one inch.—Those lines can be seen only by the assistance of a very good microscope.

Another workman has drawn a silver wire, the diameter of which does not exceed the 750th part of an inch.

But those prodigies of human ingenuity will appear extremely gross and rude, if they be compared with the immense subtilty of matter which may every where be observed amongst the works of nature. The animal, the vegetable, and even the mineral, kingdom, furnish numerous examples of this sort.

What must be the tenuity of the odoriferous parts of musk, when we find that a piece of it will scent
a whole

a whole room in a short time, and yet it will hardly lose any sensible part of its weight. But supposing it to have lost one hundredth part of a grain weight, when this small quantity is divided and dispersed through the whole room, it must so expand itself as not to leave an inch square of space where the sense of smell may not be affected by some of its particles. How small must then be the weight and size of one of those particles?

The human eye, unassisted by glasses, can frequently perceive insects so small as to be barely discernible. The least reflection must shew him, that the limbs, the vessels, and other necessary parts of such animals, must infinitely exceed in fineness every endeavour of human art. But the microscope has discovered wonders, that are vastly superior, and such indeed as were utterly unknown to our forefathers, before the invention of that noble instrument.

Insects have been discovered, so small as not to exceed the 10000th part of an inch: so that 1000000000000 of them might be contained within the space of one cubic inch; yet each animalcule must consist of parts connected with each other; with vessels, with fluids, and with organs necessary for its motions, for its increase, for its propagation, &c. How inconceivably small must those organs be? and yet they are unquestionably composed of other parts still smaller, and still farther removed from the perception of our senses.

We

We might easily fill a great many pages with examples and calculations relative to this subject; but as the pleasing narration of such wonderful facts is not likely to give any real information concerning the general properties of matter, which form the subject of this part of the book, I must refer the inquisitive reader to other works*. The consideration of this divisibility does also lead the mind to certain curious speculations. (1)

The

* Boyle's book of Effluvia; Keill's Introduction to Nat. Phil.; Rohault's Physicks; Phil. Trans. N. 194; s'Gravefand's Phil.; Musschenbroek's Phil. &c.

(1.) Several writers, when treating of the divisibility of matter, have mentioned two curious theorems, which I shall subjoin in this note, as they may be of use to the speculative philosopher. Those theorems are established on the supposition that matter is divisible without end.

Theorem I. A quantity of matter however small, and any finite space however large, being given; it is possible that that matter may be diffused through all that space, and so fill it, as not to leave in it a pore, whose diameter will exceed a given right line.

Let the given space be a cube, whose side is AB, fig. 2d. P. 1. so that the cube be equal to \overline{AB}^3 , and let the quantity of matter be represented by b^3 ; also let the line D be the limit of the diameter of the pores.

The side AB being a finite quantity, may be conceived to be divisible into parts equal to the line D. Let the number of those parts be represented by n , so that $nD = AB$, and $n^3 D^3 = \overline{AB}^3$. Conceive the given space to be divided into cubes, each of whose sides be equal to the right line D, and

The contemplation of those wonders of nature, cannot fail of impressing on our minds a strong idea of humility as well as of astonishment.—A vast gradation of animals perfect in their kind, but smaller than the human being in size and duration, descends as far down as our eyes can possibly discern, even when they are assisted by the most powerful microscopes. This vast gradation, instead of exhausting the powers of nature, shews the probable

D, and the number of those cubes will be n^3 , which cubes are represented in the fig. by E, F, G, H. Again, let the particle b^3 be supposed to be divided into parts whose number be n^3 ; and in each cubic space let there be placed one of those particles; by which means the matter b^3 will be diffused through all the given space. Besides each particle being placed in its cell, may be formed into a concave sphere, whose diameter may be equal to the given line D; whence it will follow, that each sphere will touch that which is next to it; and thus the quantity of matter b^3 , be it ever so small, will fill the given finite space, however large, in such a manner as not to leave in it a pore larger in diameter than the given line D.

Corollary. There may be a given body, whose matter if it be reduced into a space absolutely full; that space may be any given part of the former magnitude.

Theorem II. *There may be two bodies equal in bulk, whose quantities of matter may be very unequal, and though they have any given ratio to each other, yet the sums of the pores or empty spaces in those bodies may almost approach the ratio of equality.*

The

bable existence of animated beings vastly smaller than those; nor have we the least reason to fix a limit to the series.

If we contemplate the state of existence of those animals; of one, for instance, out of a large number of the same species, that has been born in a glass of dirty water; whose life lasts but a few hours, and whose size is less than the 5000th part of an inch; for such animals have been actually seen. If we indulge our fancy by considering what knowledge, or what ideas, can he possibly entertain of man, — of the earth, — of the universe; we may without difficulty conclude, that, far from having any precise notions of our existence, he may in all probability

The demonstration of this theorem is easily derived from the foregoing, for since the matter of a body may be conceived to be condensable into any part of the original bulk; therefore supposing two bodies, A and B of equal bulk, to be such that the matter of A be 100 times the matter of B; the matter of B may be conceived to be condensed into one 1000000th part of its original bulk, and of course the matter of A will be condensed in one hundred 1000000th parts of the same bulk; in which case the spaces left in the original bulk of B will be to the spaces left in the original bulk of A as 999999 to 999900, which numbers are nearly equal to each other.

Instead of the above-mentioned numbers, the proportion of the quantities of matter may be increased at pleasure, and so may the proportion of the original bulks of the bodies to the spaces into which they may be conceived to be condensable.

probability look upon the glass of dirty water as the boundary of the habitable world. Out of that water, tradition or his own experience, shews him nothing but the inevitable destruction of his species, and a confused assemblage of immense objects, whose nature and whose motions are utterly inexplicable to him. Yet he may possibly suspect that those very objects have powers infinitely superior to those of his own species.

Let us now follow the analogy, and let us briefly apply the same contemplation to ourselves. The planets, the stars, the comets, and perhaps an infinity of other bodies that are far beyond the reach of our knowledge, manifest the existence of powers infinitely above us, and perhaps even less comprehensible to us than we are to the above-mentioned animalcule. Confined to the globe of this earth, which is only a speck in the universe; and, with respect to us, not much better nor worse than the glass of dirty water is with respect to those insects; how insignificant are our powers, and how imperfect is our knowledge of nature! How little likely are we to comprehend the real order of things, and the Great Wisdom that regulates the whole! In this sublime inquiry the assistance of our reasoning faculty is trifling indeed; the clew of analogy is short and imperfect; and our imagination soon loses itself in the boundless extent of immensity.

Impenetrability is that property, by which a body excludes every other body from the place which itself

itself occupies. Thus one cannot drive a cubic inch of gold into a cubic inch of silver. You may indeed melt and incorporate the two metals into one lump; but then the lump will measure two cubic inches; which proves not that the gold occupies the same cubic inch of space which is occupied by the silver; but that the particles of the two metals are placed contiguous to each other. Thus also, if a quantity of water be put into a strong vessel, for instance, of iron, and the vessel be accurately shut up, it will not be possible to press the sides of the vessel towards each other; the matter which fills the cavity of it being sufficient to resist any degree of pressure.

Though impenetrability be admitted as a general property of matter, it must, however, be observed, that in certain mixtures of two or more bodies of different natures, a loss of bulk does actually take place; thus if a cubic inch of spirit of wine be mixed with a cubic inch of water, the bulk of the mixture will be somewhat less than two cubic inches; yet the weight of the mixture (provided no evaporation be allowed to take place) will be equal to the sum of the weights of the two fluids; which indicates that one of the fluids must have filled up some of the pores or vacuities of the other fluid. It is besides not unlikely that some other finer fluid may have escaped in the act of mixing the two bodies.

In other parts of this work we shall take notice

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of the loss of weight and other phenomena, that take place in many cases of mixture ; but with respect to impenetrability itself, we may rather consider it abstractly as a property of the real quantity of matter which exists in bodies, independently of pores and vacuities, than as a general property, without exception, of bodies in their usual state of existence.

Mobility of matter is that essential and general property, whereby any body is capable of being moved from one part of absolute space to another part of it. Experience constantly shews, that the force, which is required to move a body, is proportionate to its weight ; therefore we conclude with saying, that all bodies are capable of being moved ; provided an adequate force be employed to put them in motion.

It is a fact proved by constant and universal experience, that the progress of a body in motion is retarded precisely in proportion to the obstruction which the body meets with in its way. Thus if two bodies, A and B, exactly alike in shape, weight, and substance, be put in motion by equal impulses, and meet with equal obstructions ; by moving, for instance, through the same medium, or by rolling over the same sort of plain surface, those two bodies will run over equal spaces in equal times ; but if the body A meets with half the obstruction that the body B meets with, then A will go as far again as the body B ; when A meets with a quarter of the obstruction,

obstruction, it will go four times as far as B ; and in short, A will percur a space longer than B, by as much as its obstruction is diminished ; and consequently when the obstruction to A's motion is entirely removed, A will go infinitely farther than B ; that is, it will continue to move for ever. It therefore appears, that a body once put in motion has no power to stop itself ; nor can its motion cease, unless some force is exerted by some external power against it.

By the same sort of reasoning, we prove that a body at rest has no power to put itself in motion, and of course that it will continue for ever at rest, unless it be impelled by some external power ; for since we find that a certain impulse is required to move a body with a certain quickness, viz. so as to let it run over the space of a mile in one minute ; that with half that impulse it will percur half a mile ; with the hundredth part of the original impulse it will percur the hundredth part of a mile ; it will naturally follow, that without any impulse at all, it will not move in the least : a body therefore has no power either to put itself in motion if it be at rest, or to stop itself if it be in motion : and this passiveness of matter is called the *vis inertiae*, or *want of activity*, of bodies.

A novice in philosophy may perhaps be induced to suspect the truth or generality of this property of matter, by observing that a man, or other animal, can easily move himself from rest, or stop his

motion : but in this case it must be remarked, that the animal receives a general impulse at the commencement of his life, and that all his actions, as long as he exists, are the consequence of that original impulse. I shall endeavour to illustrate this matter by an instance of a much less complicated nature.

It is very well known that a common eight-day clock, when it is once wound up, will continue to move its pendulum for a whole week, and at the end of every hour it will strike a number of strokes on the bell. It is evident likewise that those motions of the pendulum, the hammer, &c. are owing to the original power or impulse which was communicated to the machine by the person who wound it up ; yet an ignorant man might say, if bodies cannot put themselves in motion, nor can they stop themselves when they are actually in motion ; how does it happen that the striking part of the clock puts itself in motion, and then stops itself at the end of every hour ? The answer is, that the power which was communicated to the spring or weight of the clock, is so regulated by the mechanism, as to act by little and little, sufficiently to keep the pendulum and the wheels in motion ; and that when a particular part of one of those wheels comes against a certain machinery, it then disengages a portion of the other power, viz. of the spring or weight of the striking part, which puts the hammer in action.

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What has been said of the clock will perhaps be sufficient to remove the difficulty respecting the apparent self-moving power of more complicated mechanisms, such as that of an animal or vegetable body. But though we are led by the analogy of much simpler movements, to admit the dependence of animal and vegetable motion on an original impulse ; we do not, however, presume to explain the origin, dependence, and possible modifications of that impulse ; our understandings, and our knowledge, being as yet insufficient to explain the nature and the laws of that original energy.

Attraction is that property whereby one body or part of matter attracts, or endeavours to get near, another body. There are several sorts of attraction ; such as the magnetic attraction, which takes place between magnets and iron ; the electric attraction, which is observed amongst bodies in certain circumstances, &c. These attractions, however, belong to certain bodies only, and of course they must be examined in other parts of this work. But there is a sort of attraction which belongs to bodies of every kind ; it is mutual among them, and it seems to pervade the universe. It is that property whereby bodies tend, or fall, towards the centre of the earth, and it has been called *gravitation*, because the quantity of that tendency in different bodies, is the measure of their *weight* or *gravity*.

Experience, reasoning, and analogy, shew that this gravitation exists not only between the globe of

the earth and the surrounding bodies, but between all parts of matter. One terrestrial body gravitates or tends towards another terrestrial body ; the moon gravitates towards the earth ; the moon, the earth, and all the planets, gravitate towards each other, and towards the sun ; and probably the sun, with all its planetary system, may gravitate towards some other object.

The motion of certain bodies which seem to fly away from the earth, must not be considered as an exception of this general law ; for in those cases the bodies only give way to other surrounding bodies of a heavier nature, viz. that have a greater tendency towards the earth. Thus smoke, when extricated from burning bodies, goes upwards, or from the centre of the earth, because the surrounding air, which is heavier than smoke, takes its place : but if the air be removed, or at least it be so far rarefied as to become lighter than smoke, then the smoke will descend like a stone or other heavier body. Thus also if you drop a piece of cork into an empty vessel, the cork will go downwards or to the bottom of the vessel ; but if afterwards you pour water into the vessel, the cork will ascend in order to make way for the water, which has a greater tendency towards the centre of the earth than an equal bulk of cork.

Daily and constant experience shews to every person, that near the surface of the earth, all bodies tend towards the centre of it, unless they are hindered by other bodies. . But the reader may naturally

turally ask, how is it known that the planets and the earth gravitate towards the sun? The answer is, that from the accurate measurements of the motions of those planets, they are found to follow the same laws that bodies do, which are projected in a certain manner near the surface of the earth, and whose motion is undoubtedly determined by the power of gravitation; we therefore, according to the rules of philosophizing, attribute similar causes to similar effects, and conclude that the planets gravitate towards the sun, in the same manner as stones, water, and other terrestrial bodies, gravitate towards the earth.

What is the cause of gravitation, or how can a body act upon another body through a certain space? is a question which naturally presents itself to the inquisitive mind; but which we are utterly incapable to answer.

A variety of conjectures have been formed, and many hypothetical suppositions have been offered, for the elucidation of this question; but as they are all involved in absurdity and obscurity, I shall not detain my reader with any account of them. All we can say is, that the effect is certain, the knowledge of its laws is highly useful to mankind; but its cause is hidden amongst the mysteries of nature.

CHAPTER IV.

OF MOTION IN GENERAL; THE LAWS OF MOTION, AND THE PROPERTIES OF SIMPLE MOTION.

OF the general properties of matter, the first three may be presumed to have been sufficiently illustrated in the preceding chapter; but the other three, viz. mobility, *vis inertiae*, and gravitation, are the foundation of the extensive doctrine of motion, or of mechanics; and are therefore deserving of a full and particular examination.

Almost all the phenomena of nature are owing to motion. The appearance and disappearance of the celestial bodies; the increase of animals and vegetables; the composition and decomposition of complex substances, fire, &c. are all effected by motion. Therefore the laws of motion must be looked upon as the foundation of natural philosophy; so that without a clear comprehension of those laws, it will be impossible to make any proficiency in the study of nature.

The importance and extent of the subject, render it necessary to divide the materials into several chapters, in each of which such particulars will be arranged, as are more immediately connected with each other, and more conducive to conciseness and perspicuity.

It

It is a natural consequence of the *vis inertiae* of matter, that whatever body is in motion, must be supposed to have been put in motion by some active force; viz. some external impulse.

This impelling force may be of two sorts. It may either communicate the impulse at first, and then cease to act, like the impulse which is given to a bullet by the discharge of a gun; or it may act irremittedly on the body in motion, like the force of gravity on a stone that is dropped from any height. For distinction sake we shall call the first simply an *impulse*, and the latter an *accelerative force*.

A body may be put in motion by one, two, or more forces at the same time, and those forces may be either all simple, or all accelerative, or some may be of one sort, and others of the other sort.

Most of the movements that commonly take place in the world, are the effect of more than one impulse; and they are never performed with perfect freedom, since they are always performed in resisting mediums. However, in order to preserve perspicuity as much as it lies in our power, we shall in the first place examine the motions arising from a simple impulse in a non-resisting medium, and shall then proceed in the examination of the more intricate causes of motion.

Three general laws of motion have been deduced from innumerable experiments and observations,
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by means of the strictest philosophical reasoning.—
They are as follows :

I. Every body will continue in its state of rest, or of moving uniformly in a straight line ; unless it be compelled to change that state by forces impressed.

II. The change of motion is always proportional to the moving force impressed, and is always made according to the right line, in which that force is impressed.

III. Action and re-action are always equal and contrary to each other ; or the actions of two bodies mutually upon each other, are always equal, and directed towards contrary parts.

The first of those laws is evidently nothing more than the *vis inertiae* of matter, announced in a different manner ; excepting only the assertion of the body moving in a straight, and not in a curve, line, which particular may perhaps be deserving of some explanation.

The proof of this particular property has likewise been deduced from constant experience ; for we find that whenever a body moves in a curve line, there always is some secondary power which forces it to deviate from the rectilinear course ; and that deviation is exactly proportional to that secondary power. Thus a stone which is thrown horizontally would proceed horizontally in a straight line, were it not drawn downwards by the
force

force of gravity; and we find by computation, that the deviation from the horizontal direction is exactly proportional to the force of gravity.

Hence the second law has been deduced, in which it is asserted that the change of motion is always proportional to the moving force impressed, and is made according to the right line in which that force is impressed; for if it were made in a crooked line, it would imply the action of a third force; and if it were not proportional to the moving force, the effect would not be adequate to the cause.

The third law may be easily illustrated by means of examples; and the least reflection on the phenomena, which commonly occur, will be sufficient to manifest the truth and universality of it.

When a man strikes one of his hands against the other, the blow is felt equally by both hands. If you strike a glass bottle with a steel hammer, the blow will be received equally by the hammer and by the glass bottle; and it is immaterial whether the hammer be moved against the bottle at rest, or the bottle be moved against the hammer at rest; yet the bottle will be broken, whereas the hammer will not, because the same blow, which is sufficient to break glass, is not sufficient to break a lump of steel.—It is for the same reason, that if a man strike his fist against another man's face, the blow, which is equally received by the fist and by
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the face, will produce a material hurt on the latter, but not upon the former.

If a stone be tied to a horse by means of a rope, the horse in dragging the stone will exert a degree of force equal to the resistance of the stone; for the rope which is stretched both ways will equally pull the horse towards the stone, and the stone towards the horse. And, in fact, the stone will not follow the horse, unless the power of the horse be greater than the resistance of the stone.

Experience likewise shews, that if a loadstone and a piece of iron be placed on separate pieces of cork, and be suffered to float on the surface of water, the attraction between them will be mutual, and they will move towards each other so as to meet in a place between their two original situations. If the loadstone only be held fast in its place, the iron will come all the way to meet it; and if the iron only be held fast in its place, the magnet will advance towards the iron until it comes in contact with it.

The motion given to a boat by oars is likewise a convincing illustration of the third law; for by the action of one extremity of each oar against the water one way, its other end re-acts upon the boat, and impels it the contrary way.

We shall now examine the motion which is produced by a single impulse, which acts at first only,
and

and then leaves the body to proceed by itself, in an unresisting medium.

It has been already shewn, that in this case the body will continue to move uniformly; that is, it would run over equal spaces in equal portions of time; and such would be the case of a bullet shot out of a gun, or of a stone thrown out by a man's hand, were they not impeded by the resistance of the air, and were they not acted upon by the force of gravity. But it is now necessary to take notice of several particulars relative to this sort of motion.

In the first place it may be asked, how does the impelling force put the body in motion, or what does it communicate to the body? The answer is, that the moving force does not communicate any thing to the body; but it only moves the body through a certain space in a certain time, after which the body, being left to itself, will continue to move at the same rate, *viz.* will continue to run over like spaces in the like portions of time; and that merely in consequence of its *vis inertiae*; of which *vis inertiae*, however, we do not pretend to know any thing more, than that it has been found to be a general property of matter.

All the particulars which can be remarked with respect to the above-mentioned simple motion, are the relations between the time, in which a certain space is described; the space which is percutred in a certain time, the quantity of which shews
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the *velocity* ; the quantity of matter in motion ; and lastly, the *momentum*, by which word we mean the force of the body in motion, and reckon it equivalent to the impression that the body in motion would make on another body at rest, that should be presented to it precisely in the direction of its motion.

The *momentum* has been often called *the quantity of motion*, or simply *the motion* ; but we shall not make use of the last word in this sense, lest it should be mistaken for the velocity, in which sense it has been likewise used. We shall also express the above-mentioned four particulars by their initial letters, viz. T for the time, S for the space, V for the velocity, Q for the quantity of matter, and M for the momentum.

By the word *velocity* we mean nothing more than the ratio of the quantity of space which is run over in a certain portion of time. Thus it is said that a body moves with the velocity of three feet per second ; also that the velocity of a body A is to the velocity of another body B, as two to three ; meaning that if A goes over a certain space, as for instance, four miles, in a certain time, the body B will percur six miles in the same time ; since two is to three as four is to six.

It is therefore evident, that in equal times the velocities are as the spaces ; but if the times be unequal, then the velocities are as the quotients of the spaces divided by the times respectively. Thus
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suppose that a body A passes over ten feet in two minutes, and another body B passes over eight feet in four minutes, the velocity of A will be to the velocity of B as $\frac{10}{2}$ to $\frac{8}{4}$; that is as five to two; for by dividing the ten feet by the two minutes, we find how many feet the body A runs over in *one minute*, and likewise by dividing the eight feet by four minutes, we find how many feet the body B runs over in the same time; viz. *one minute*; so that by the operation of dividing the spaces by the times respectively, we do nothing more than find out the spaces that are percurred by the two bodies in *equal times*, and then compare them together.

Before we proceed any farther, it is necessary to observe, that whenever it is said that certain things are as certain other things, we only assert the ratio of the former to the latter; viz. that the former increase or decrease according as the latter do increase or decrease; but from such assertions nothing real and determinate can be deduced, unless we have recourse to experiments, in order to ascertain some of those particular things with which others are compared. Thus in the preceding paragraph, it has been asserted that the velocities are as the quotients of the spaces divided by the times; yet this assertion will not enable us to determine the velocity, or the space run over, or the time, which is employed by a certain body in motion, unless

unless some of those particulars be previously known. Hence if we learn from actual experiment (viz. by measuring the space with a ruler and the time by a watch), or are otherwise informed, that a body has been moving through ten feet in two seconds; then dividing the ten by two, the quotient five gives the velocity; which means that the body moves at the rate of, or percurs, five feet per second. If by the above-mentioned proportional expression we wish to find the space, we must previously know the velocity and the time; and if we wish to ascertain the time, we must previously know the velocity and the space. Therefore, in general, the use of such proportional expressions is to render certain particulars deducible, by computation, from other particulars which belong to the same expression, and which have been previously ascertained by means of actual experiments. We shall now proceed to explain the other particulars which relate to the above-mentioned simple or equable motion.

The *space* is as the velocity multiplied by the time; (that is, S is as $V T$) for if a body move with the velocity of three feet per minute, it is evident that it must pass over twice three, or six, feet, in two minutes; three times three, or nine, feet, in three minutes; four times three, or twelve feet, in four minutes; and, in short, the space is as the product of the velocity, or rate of going, multiplied by the time.

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The *time* is as the space divided by the velocity; (viz. T is as $\frac{S}{V}$) for if a body, for instance, runs over 12 feet when its velocity is three feet per minute, it is evident that in order to find the number of minutes, that the body has employed in passing over 12 feet of space, we must say, by the common rule of three, if the body passes over three feet in one minute, how many minutes will it employ in passing over 12 feet; which proportion is stated thus; $3 : 1 :: 12 :$, and as the second term is unity, we need only divide the 12 by 3; (viz. the space by the velocity) and the quotient 4 is the time sought.

The *momentum*, and the *quantity of matter*, are the two last particulars which remain to be examined with respect to this sort of motion. It has already been mentioned, that the momentum is the force of the body in motion, and is equivalent to the impression it would make on another body that should be placed at rest directly before it.

According to the fourth axiom, every effect must be produced by an adequate cause; therefore if a body be caused to move with a certain velocity by means of a certain impulse, the double of that impulse will be required to make it move with the double of that velocity; three times that impulse to let it move with three times the original velocity;

city; and, in short, the moving force or impulse must be proportionate to the velocity. And for the same reason, the resistance, which must be opposed to the said body in order to stop it, must likewise be proportionate to the velocity of the body.

Now let two distinct bodies, A and B, move with equal velocities; but let the quantity of matter in B be the double of the quantity of matter in A; and it is evident that the momentum of B must be double the momentum of A; for if we imagine B to be divided into two equal parts, each of those parts must have a momentum equal to the momentum of A; (A being equal to the half of B) and of course both halves together must have a momentum double of the momentum of A.

If the body B be supposed to move as fast again as A, or with the double of its former velocity, it follows, from what has been mentioned above, that its momentum must be double of its former momentum; but before its momentum was double the momentum of A, therefore now its momentum must be quadruple the momentum of A; that is, it must be multiplied by two on account of its double quantity of matter, and again by two on account of its double velocity; which is as much as to say that the momentum is as the product of the quantity of matter multiplied by the
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the velocity; (viz. M is as $Q V$.)—Or we may consider it as a definition, and say that by the *momentum* we mean the product of the quantity of matter by the velocity*.

If the quantity of matter in B , instead of being double, be supposed to be treble, or quadruple, or the half, or other multiple, of the quantity of matter in A ; the same mode of reasoning will shew that its momentum must be treble, or quadruple, or the half, or any other multiple respectively of the momentum of A , when the velocities of A and B are equal; but that those momentums must be multiplied by the velocities when the velocities of the bodies A and B are unequal; which proves that the proposition is universally true.

* The measure of the momentums of bodies, under the title of *vis matrix*, or *vis viva*, when moving with different velocities, produced some years ago a long and loud dispute amongst the learned in Europe. The intricacy of the arguments would render a statement of the question too long for this work, and it would besides be attended with little or no profit to the beginner; I shall therefore refer such of my readers as are desirous of being informed relatively to this question, to two excellent tracts; the first of which is entitled *An Essay on Quantity*, by the Reverend Mr. Reid, in the 45th vol. of the *Phil. Transf.* the second is *An Inquiry into the Measure of the Force of Bodies in Motion*, by Dr. Irwin, *Phil. Transf.* for 1745.

Lastly, the quantity of matter is as the momentum divided by the velocity; (viz. Q is as $\frac{M}{V}$); for let V in the preceding proportional expression (M as V Q) be represented by the number 2; then that proportional expression will become M as 2 Q ; meaning that the momentum is as twice the quantity of matter; but if the momentum is as twice the quantity of matter, therefore, taking the halves of those quantities, (for the halves, or the quarters, or any other like parts, or multiples of two quantities, have the same proportion to each other as the quantities themselves. Euclid. Elem. B. v. prop. 15.) half the momentum will be as the quantity of matter, which is expressed thus; Q is as $\frac{M}{2}$. Again, if the velocity be represented by any other number, as by 12, the proportion M as V Q , will become M as 12 Q , and, taking the 12th part of those two quantities, we say, that since the momentum is as 12 times the quantity of matter, therefore the 12th part of the momentum is as the quantity of matter, which is expressed thus; Q as $\frac{M}{12}$; but the velocity is represented by the number 12 in the last supposition; by the number 2 in the preceding supposition, and may be represented by any other number; therefore, universally, the quantity of matter is as the quotient of the momentum divided by the velocity.

I shall

I shall now collect all the propositions, or laws, which belong to simple motion, under one point of view, and, for the sake of perspicuity, I shall express them both in the concise way, by using the initial letters, and in words.

V is as $\frac{S}{T}$; S is as V T; T is as $\frac{S}{V}$; M is as V Q; and Q is as $\frac{M}{V}$.

The same expressed in words.—In *simple motion*, viz. when a body is put in motion by a single impulse, which acts at first, and then leaves the body to proceed by itself in a non-resisting medium; or when several bodies are thus separately put in motion; the velocities are as the spaces divided by the times; the spaces are as the velocities multiplied by the times; the times are as the spaces divided by the velocities; the momentums are as the velocities multiplied by the quantities of matter; and, lastly, the quantities of matter are as the momentums divided by the velocities.

Thus, considering the importance of the subject, I have endeavoured to demonstrate the particulars relative to simple motion, in as familiar a manner, and as little encumbered with mathematical expressions, as the subject seemed to admit, purposely to adapt them to the capacity of beginners. And I must earnestly entreat the reader to make himself master of the contents of this chapter before he proceeds to the next.

CHAPTER V.

OF THE MOTION ARISING FROM CENTRIPETAL,
AND CENTRIFUGAL, FORCES; AND OF THE
CENTRE OF GRAVITY.

A *Centripetal force* is that power which compels bodies to move, or to tend towards a point, which is called the *centre of attraction*. A *centrifugal force*, on the contrary, is that power which compels bodies to recede from a point, which is called the *centre of repulsion*. *Gravitation*, or that power, by which bodies are forced to fall towards the centre of the earth, is a centripetal force, and will serve us as an example for the illustration of the general theory.

But though bodies direct their course towards the centre of the earth, yet the attractive power must not be considered as a peculiar property of that centre, or of any particular body near it. Attraction is a property which belongs to matter in general, and is proportionate to the quantity of it. The parts of the earth mutually gravitate towards, or attract, each other;—a stone attracts another stone, or any other body; the earth attracts a stone, as well as the latter attracts the former, and all bodies, in short, mutually attract each other; nor are we acquainted with any particle of matter which may
be

be said to be destitute of attraction towards the whole assemblage of terrestrial bodies. That, *ceteris paribus*, the attractive force is proportionate to the quantity of matter, may be easily proved; for let A, B, and C be three bodies equal in every respect; and if A attract C with a certain force, (for instance, a force equal to one ounce) it is evident that B, its equal, must likewise attract C with the force of one ounce; and, of course, A and B together, or a body equal to those two, must attract C with the force of two ounces. Again, if we take ten equal bodies, it is evident, that two of them will attract another distinct body with twice the force of one of them only, as also that four, or five, or six of those equal bodies will attract the other body with four, or five, or six times respectively the force of one of them only, and so forth; which evidently shews the generality of the proposition.

It is in consequence of this truth, that when a body A prevents another body B from falling towards the centre of the earth, the former is pressed by the latter, and that pressure is proportionate to the quantity of matter in B. Now, that pressure is called the *weight* of the body B, and the quantity of it is expressed by comparing it with a certain arbitrary standard weight, which may be called an ounce, a pound, a grain, &c. So that when a certain body A is said to weigh three pounds, whilst another body B weighs one pound,

the meaning is, that the quantity of matter in A, and of course its attraction towards the earth, is treble the quantity of matter in B, or the attraction of B towards the earth.

Since attraction is a general property of matter, it may be asked, why do we not perceive any attraction between the bodies which usually surround us, as for instance between two flints, or two pieces of lead? The answer is, that the attractive force of matter in general is too small to become perceptible, excepting when the bodies, or one of them, is very large, as is the case between the earth and a flint, or other body; for if you suppose that a flint stone A be equal to the $\frac{1}{100000000000000}$ part of the whole earth, and likewise suppose that another body B is attracted by the earth with a force equivalent to one pound; then it follows that the body B must be attracted by the flint stone A with a force equivalent to the $\frac{1}{100000000000000}$ part of a pound; which is too small to produce any sensible effect. Yet, notwithstanding this, the accuracy and improvements of the present age, have found means of rendering the attraction between bodies of no great size, sufficiently sensible; but the account of such experiments will be found in another part of this work.

Considering that the attraction is mutual between bodies, as between a stone and the earth, it may be asked, why does not the earth move to-

wards the stone at the same time that the stone moves towards the earth? The answer is, that the earth, agreeably to the theory, must actually move towards the stone, but its motion is too small to be perceived by our senses; for if we suppose that the earth is at least $1000000000000000000000000$ times larger than the stone, the attraction of the earth for the stone, must be to the attraction of the latter for the former, as that immense number is to unity. Now since the effects are always proportionate to their causes, it follows, that if in a certain time the stone moves through 1000 feet in its descent towards the earth, the earth must in the same time move towards the stone through

$$\frac{1000}{1000000000000000000000000} \text{ parts of a foot;}$$

or (which is the same thing) through the $1000000000000000000000000$ th part of a foot; a quantity vastly too small for our perception.

Were the two bodies not so disproportionate, they would both be seen to move towards each other. Thus if two equal bodies, as A and B fig. 3. Plate I. be placed at a certain distance of each other, and be then left at liberty, viz. free from any obstruction, they will move towards each other, and will meet at a point C midway between their original situations. But if the bodies be unequal; for instance A in fig. 4. Plate I. be three times as big as B, then they will meet at a point C, which is as much nearer the original situation of A, than that of B, as the body A is bigger

bigger than the body B; viz. AC will be equal to one third part of BC; for since the quantity of matter in A is equal to three times the quantity of matter in B, the attraction of the former must be three times as great as the attraction of the latter, consequently the space run over by the body B must be three times as great as the space run over by the body A, in the same time.

It is evident that the like reasoning may be applied to bodies that bear any proportion to each other; hence we conclude that *the distances of the original situations of the bodies from the point C, where if left at liberty they will meet in consequence of their mutual attraction, are inversely as their quantities of matter*; viz. as the quantity of matter in A, is to the quantity of matter in B, so is the distance BC, to the distance AC.

The point C is called *the centre of gravity* of those two bodies; being in fact the point, or centre, towards which they gravitate, and where they will actually meet, if not disturbed by any external force or impediment.

What has been observed with respect to the two bodies, may be easily applied to the mutual attraction of three, or four, or, in short, of any number of bodies; there being always a centre of gravity which is common to them all. Such also is the case with a single body; viz. there is a point in any single body, which is its centre of gravity, towards which, if the body were divided into different

ferent parts, those parts would gravitate. The nature and properties of the centre of gravity will be farther noticed in the next chapter.

Since the attractive power is proportionate to the quantity of matter, it follows, *that all sorts of bodies, however different they may be in their weights, if they begin to move towards the earth from the same height, at the same time; they must be equally accelerated; that is, they must all descend through the like space in the same portion of time; for though a body A be twice as heavy as another body B, if you imagine that the former is divided into two equal parts, each of those parts must be equal to B, and of course it must move through an equal space, as B, in the same time.* Now it is evident, that when the two parts of A are joined together, the effect must be the same. The like reasoning may be extended to bodies, whose quantities of matter bear any other proportion to each other. Hence all sorts of bodies, when left at liberty, would fall from the same height to the ground precisely in the same time, were they not unequally resisted by the air through which they move. I say unequally resisted, because that resistance is in proportion not to the quantity of matter, but to the surface, when the quantities of matter are equal. This may be satisfactorily proved by a variety of experiments. Take, for instance, a small quantity of cotton, spread it as much as you can, then let it fall from your hand to the ground, and you will find that
the

the cotton will employ three, four, or more, seconds of time in that descent. But if you take up that cotton and compress it into a very small compass, you will find that on repeating the experiment, the same quantity of cotton will descend to the ground in less than a second. Thus also if you drop from the same height at the same time a guinea and a common gold leaf, the guinea will come to the ground incomparably quicker than the gold leaf. But if you compress the leaf so as to form it into a small lump, and repeat the experiment with this lump and the guinea, they will be found to touch the ground nearly at the same moment *.

The converse of the last proposition is likewise evident ; namely, *that if bodies, in falling from the same height towards the centre of the earth, describe equal spaces in the same portion of time, the attraction*

* This proposition is confirmed in a manner less easy indeed, but more evident and conclusive, by means of a tall glass receiver, having a mechanism at its upper end, from which a guinea and a feather, or other light body, may be dropped at the same time. When this glass receiver is set straight up, and is exhausted of air, in the manner which will be described hereafter, the above-mentioned guinea and feather, will, on being disengaged, arrive at the bottom of the receiver at the same moment precisely. But if the receiver be not well exhausted of air, then the feather will arrive at the bottom later than the guinea ; and much more so when the receiver is quite full of air.

must be proportionate to their quantities of matter; otherwise the spaces, &c. would not be equal.

Hitherto we have taken notice of the properties which naturally arise from the attraction being proportionate to the quantity of matter. It is now necessary to examine the actual motion of bodies which move towards a centre of attraction.

The great difference between the simple impulse, mentioned in the preceding chapter, and a centripetal, or centrifugal, force, is that the former produces equable motion; that is, such as compels bodies to describe equal spaces in equal portions of time; whilst the latter produces unequable motion; viz. it compels bodies to describe unequal spaces in equal portions of time.

This inequality arises from the continual action of the latter power; for a centripetal, or centrifugal, force, does not act at first only; but it does continually act upon, and impel, the bodies in motion; that is, the centripetal, towards the centre of attraction, and the centrifugal, from the centre of repulsion.

The attraction of the earth, or gravitating power, has been found, from a variety of facts, which will be mentioned hereafter, to decrease in proportion as the squares of the distances from the centre of the earth, increase; or, in other words, the force of gravity at different heights is inversely as the squares of the distances from the centre of the earth. At a height, for instance, as far from
the

the surface of the earth as the surface is from the centre, the force of gravity is a quarter of what it is at the surface; for the distances being as one to two, their squares are one and four; therefore, as one is to four, so is the force of gravity at the above-mentioned height, to the force of gravity at the surface.

This diminution of intensity in the proportion of the squares of the distances from the centre of emanation, seems to take place not only with the force of gravity, but likewise with all sorts of emanations from a centre, such as light, sound, &c. as far however as we are able to judge from the present state of knowledge; for with the decrease either of sound or of light, this law has not been ascertained to any great degree of accuracy.

But, independently of actual experiments, it may be strictly demonstrated, that *emanations, which proceed in straight lines from a centre, and do not meet with any obstruction, must decrease in intensity inversely as the squares of the distances from the centre.* (1)

Bodies

(1) Let A, fig. 5. Plate I. be the centre of emanation (for instance the flame of a candle.) Let OPBv be a square hole, and drawing straight lines from A to the corners of this square, produce them indefinitely towards J, H, E, &c.

In the first place it is evident that the light which passes through the square hole OPBv, will fill all the space between

Bodies that are left to fall from any height, will move faster and faster the nearer they come to the surface of the earth; for if the force of gravity acted upon a body only at the commencement of

tween the four straight lines AH, AI, Ar, and AE. Secondly, it is also evident that if a plane surface be placed at E, parallel to the square OPBv, all that part of it which lies between the aforesaid straight lines; viz. IHEr, will be illuminated by the light which passes through OPBv; but as the plane IHEr is larger than OPBv, the light upon it cannot be so dense as at OPBv; and for the same reason, if a plane be situated at D, parallel to OPBv, the light upon it will be less dense than at OPBv, but more dense than at IHEr, &c. Thirdly, it is also evident that the planes IHEr, KGDs, LFCx, are square figures, since the hole OPBv has been supposed to be a square. Therefore, the only thing which remains to be proved, is, that if the distance AC be equal to twice the distance AB, the area of the square LFCx is four times as large as the area OPBv; that if AD be equal to three times AB, the area KGDs is nine times as large as OPBv; or, in short, that the areas OB, LC, KD, &c. are as the squares of the distances from A, which is easily done; for ABP, ACF, being equiangular triangles (Eucl. p. 29. B. I.) we have (Eucl. p. 4. B. VI.) $AB : AC :: PB : FC$; but PB and FC are the homologous sides of the similar plane figures OPBv, LFCx; and (Eucl. p. 20. B. VI.) those figures are as the squares, or in the duplicate proportion, of their homologous sides; therefore $OPBv : LFCx :: \overline{PB}^2 : \overline{FC}^2 :: \overline{AB}^2 : \overline{AC}^2$. And the like reasoning may be applied to the other squares KGDs, &c.

its

its descent, the body would, (according to the laws of simple motion, Chap. IV.) continue to describe equal spaces in equal portions of time. But the very next moment the force of gravity impels the body again, in consequence of which the body's velocity must be doubled; since the second impulse is equal to the first, and the first remains unaltered. For the same reason on the third moment the body's velocity will be trebled, and so on. Or, speaking more properly, the velocity will increase as the time increases, viz. *the velocity will be as the time*; the meaning of which is, that the velocity at the end of two seconds is to the velocity at the end of three seconds, as two to three; or the velocity at the end of one minute is to the velocity at the end of one hour, as one is to sixty, &c. *.

The spaces described by such descending bodies cannot be proportionate simply to the times of descent; for that would be the case if the velocity remained unaltered; but, the velocity increasing

* The velocities are as the times when the gravitating power remains unaltered, or with the same gravitating power; but if two distinct gravitating powers be compared together, then the velocities will be as the products of the times multiplied by the gravitating forces respectively; it being evident that a double force will produce a double effect, a treble force will produce a treble effect, &c. Hence when the times are equal, or in the same time, the velocities are as the gravitating, or the impelling, forces.

continually,

continually, it is evident *that the spaces must be as the times multiplied by the velocities*; for a double velocity will force the body to move through a double space in an equal portion of time, and through a quadruple space in twice that time; also a quadruple velocity will force the body to move through a quadruple space in an equal portion of time, and through eight times that space in twice that time; and so on in any proportion. But it has been shewn above that the velocities are as the times; therefore to say that the spaces are as the times multiplied by the velocities, is the same thing as to say that the spaces are as the times multiplied by the times, or as the squares of the times; and for the same reason it is the same thing as to say that the spaces are as the velocities multiplied by the velocities, or as the squares of the velocities *.

This property of descending bodies, (viz. that they run through spaces which are as the squares of the times) has been usually demonstrated in a different way by the philosophical writers. Their demonstration may, perhaps, appear more satisfactory than that of the preceding paragraphs to some of my readers; I shall therefore subjoin it, especially as it proves at the same time another law relative to the velocity of descending bodies.

* Therefore in equal times the spaces are as the impelling, or gravitating, forces. See the last note.

Let AB , fig. 6. Plate I. represent the time, during which a body is descending, and let BC represent the velocity acquired at the end of that time. Complete the triangle ABC , and the parallelogram $ABCD$. Also suppose the time to be divided into innumerable particles, ei , im , mp , po , &c. and draw ef , ik , mn , &c. all parallel to the base BC . Then, since the velocity of the descending body has been gradually increasing from the commencement of the motion, and BC represents the ultimate velocity; therefore the parallel lines ef , ik , mn , &c. will represent the velocities at the ends of the respective times Ae , Ai , Am , &c. Moreover, since the velocity during an indefinitely small particle of time, may be considered as uniform; therefore the right line ef will be as the velocity of the body in the indefinitely small particle of time ei ; ik will be as the velocity in the particle of time im , and so forth. Now the space passed over in any time with any velocity is as the velocity multiplied by the time; viz. as the rectangle under that time and velocity; hence the space passed over in the time ei with the velocity ef , will be as the rectangle if ; the space passed over in the time im with the velocity ik , will be as the rectangle mk ; the space passed over in the time mp with the velocity mn , will be as the rectangle pn , and so on. Therefore the space passed over in the sum of all those times, will be as the sum of all those rectangles. But since the particles of
time

time are infinitely small, the sum of all the rectangles will be equal to the triangle ABC. Now since the space passed over by a moving body in the time AB with a uniform velocity BC, is as the rectangle ABCD, (viz. as the time multiplied by the velocity) and this rectangle is equal to twice the triangle ABC (Eucl. p. 31. B. I.) therefore the space passed over in a given time by a body falling from rest, is equal to half the space passed over in the same time with an uniform velocity, equal to that which is acquired by the descending body at the end of its fall.

Since the space run over by a falling body in the time represented by AB, fig. 7. Plate I. with the velocity BC is as the triangular ABC, and the space run over in any other time AD, and velocity DE, is represented by the triangle ADE; those spaces must be as the squares of the times AB, AD; for the similar triangles ABC, and ADE, are as the squares of their homologous sides, viz. ABC is to ADE as the square of AB is to the square of AD, (Eucl. p. 29. B. VI.)

In fig. the 8th. Plate I. the spaces, which are described by descending bodies in successive equal portions of time, are represented, for the purpose of impressing with greater efficacy on the mind of the reader, the principal law of gravitation. The line AB represents the path of a body, which is let fall from A, and descends towards the ground at B. The divisions on the line AB denote the places of

the body at the end of one second, two seconds, &c. which equal portions of time are marked on the left hand side ; whilst the numbers on the right express the feet percurred, or real distances from A to the first division, from A to the second division, and so on. It appears, therefore, that in one second the body has descended through 16,087 feet, that in two seconds it has descended through four times 16,087, or 64,348 feet, &c.

It may also be observed, that the spaces run through during each single second, are as the odd numbers 1, 3, 5, 7, &c. ; that is, if the space percurred in the first *second* be called one, the space percurred during the second *second* only will be three times as great, the space percurred in the third *second* will be five times as great, and so on. In fact, if we subtract 16,087 from 64,348, the remainder, 48,261, is equal to three times 16,087 ; if we subtract 64,348 from 144,783, the remainder, 80,435, is equal to five times 16,087, &c.

It has been shewn above that the force of gravity at equal distances from the centre of the earth is proportionate to the quantity of matter ; but it must be observed, that when the distances are unequal, then the gravitating forces, or weights, of bodies, are as the quotients of the quantities of matter divided by the squares of the distances respectively, or, which is the same thing, the weights of bodies are said to be as the respective quantities of matter directly, and the squares of the respective distances
inversely ;

inversely ; since the gravitating force has been shewn to decrease inversely as the squares of the distances from the centre of attraction. Thus if a body A, which is five times as big as another body B, is situated at the distance of 4000 miles from the centre of the earth, whilst B is situated at 6000 miles distance, then the weight of A will be to the weight of B as the quotient of five divided by the square of 4000, is to the quotient of one divided by the square of 6000 ; viz. as $\frac{5}{16000000}$ is to

$$\frac{1}{36000000} *$$

In

* Suppose it be required to find how much a leaden ball, which on the surface of the earth weighs twenty pounds, will weigh at the top of a mountain which is three miles high.

The semidiameter of the earth is known to be about 3985 miles, to which we add the height of the mountain, viz. three miles, and we have the two distances ; that is from the centre of the earth to the surface, 3985 miles, and from that centre to the top of the mountain 3988 miles. The squares of those numbers are 15880225 and 15904144. Then say as 15904144 is to 15880225, so is twenty pounds to a fourth proportional, which by the common rule of three (viz. by multiplying 15880225 by 20, and dividing the product by 15904144) will be found to be 19,969 ; or 19 pounds and $15\frac{1}{2}$ ounces, which is the weight of the leaden ball at the top of the mountain, viz. nearly half an ounce less than on the surface of the earth.

It must not, however, be imagined that the leaden ball, which is balanced by a counterpoise of twenty pounds in

In the preceding explanations and examples, the spaces and velocities of descending bodies have been calculated on the supposition that the force acts uniformly; viz. that during the descent of the body from A, fig. 8. Plate I. towards the ground, the attraction of the earth does not increase; which supposition, strictly speaking, is not true; for it has already been shewn, that the force of gravity decreases inversely as the squares of the distances from the centre of the earth; so that the nearer the body comes to the ground, the stronger its gravitation will be. However, in short distances from the surface of the earth, that increase of gravity is so very trifling, that for common purposes it may be safely neglected. But as the same theory is applicable to all sorts of gravitating powers, and as very great distances may sometimes enter the calculation, it will be proper to subjoin the method of calculating the velocities which are acquired by bodies descending towards a
centre

a pair of scales on the surface of the earth, will appear lighter at the top of the mountain; for this will not be the case, because the counterpoise itself will lose an equal portion of its weight by being situated on the top of the mountain; and of course the equilibrium of the scales will not be disturbed. But if the leaden ball in question be weighed in one of those weighing instruments which are made with a spiral steel spring, then indeed the decrease of its weight at the top of the mountain will be clearly perceived, provided the weighing instrument be sufficiently accurate.

centre of attraction, when the increase of the attractive power is taken into the account (2).

It is evident that, since the continual action of the force of gravity accelerates the motion of a descending body, it must continually retard the motion

(2) This problem is taken from Dr. Saunderson's Method of Fluxions.

PROBLEM.

Let S. fig. 9. Plate I. be the centre of the earth, B any point in its surface, and let the force of gravity in all places be reciprocally as the squares of their distances from the centre of the earth: *it is required to determine the velocity of a heavy body at the surface of the earth, which it acquires in falling from any given altitude AB.*

Let x be any indeterminate distance from the centre of the earth, and let v be the velocity of the falling body at that distance. Let $\frac{1}{SB^2}$ represent the force of gravity at B, and consequently $\frac{1}{xx}$ its force at the distance x . First then it is plain, that after the falling body is arrived at the distance x , and then descends further through any infinitely small space as \dot{x} , the time of that infinitely small descent will be as $\frac{\dot{x}}{v}$; that is, it will be as the space directly, and as the velocity inversely; and the infinitely small acquisition made by the velocity in that descent will be as the time $\frac{\dot{x}}{v}$ and the accelerating force $\frac{1}{xx}$ jointly; that is, \dot{v} will be as $\frac{\dot{x}}{vxx}$; therefore $v\dot{v}$ will be as $\frac{\dot{x}}{\dot{x}x}$; therefore the fluents of these fluxions, which are generated in equal times,

motion of an ascending body. A ball, for instance, which is projected upwards, will be gradually retarded by the gravitating force, which acts in a contrary direction.

The foregoing explanations relatively to the laws of gravity, or of a centripetal force, may be easily applied to the explanation of the properties of a repulsive, or centrifugal, force; for in fact the same reasoning

times, will be proportionable; that is, $\frac{1}{2}vv$ will be as $\frac{1}{SB} - \frac{1}{SA}$, or as $\frac{SA - SB}{SB \times SA}$, or as $\frac{AB}{SB \times SA}$: therefore, since the quantities 2 and SB are constant, vv will be as $\frac{AB}{AS}$.

This being discovered, let DB be the height from which a body will fall to the surface of the earth in one second of time; and since during so small a descent, the force of gravity may be looked upon as uniform, it is evident that a body falling from D to B will acquire a velocity which will carry it uniformly through the space 2BD in a second of time. Let 2BD represent this velocity; then must every other velocity be represented by the space through which it will carry a body in a second of time. Now to find the velocity acquired in falling from A to B, I say as $\frac{DB}{DS}$ is to $\frac{AB}{AS}$, so is $4DB^2$ (the square of the velocity acquired in falling from D to B) to $4DB \times DS \times \frac{AB}{AS}$, (the square of the velocity acquired in falling from A to B) $= 4mm \times \frac{AB}{AS}$, supposing m to be a mean proportional between DB and DS. Therefore a body falling from A to B acquires a velocity

reasoning will do for the one as for the other, changing only the word *attraction* for *repulsion*, and the word *acceleration* for *retardation*. Thus the velocity of a body which is receding from a centre of repulsion, is retarded in proportion as the time increases ; or the velocities are said to be inversely as the times. Also the spaces decrease, or are, inversely, as the squares of the times.

If a body be thrown perpendicularly upwards, that is, in a direction from the centre of the earth, with the velocity which it acquired by falling in a given time, it will arrive at the same height from

velocity that would carry it through the space $2m \times \frac{AB^{\frac{1}{2}}}{AS^{\frac{1}{2}}}$ in a second of time.

Scholium. DB and DS having been ascertained by means of experiments, $2m$ is thereby found to be about seven English miles.

Corollary 1. If AB be infinite, the quantity $\frac{AB}{AS}$ becomes equal to one, hence it goes out of the question ; and therefore a body falling from an infinite height, will acquire at the surface of the earth but a finite velocity ; viz. such a velocity as will carry it uniformly through seven miles in one second of time.

Coroll. 2. Therefore, *e converso*, if a body be thrown upwards with such a velocity, it will never return, but it will ascend for ever.

Coroll. 3. After the same manner we may determine the velocity of a falling body, whatever be the law of gravity.

which

which it fell, in the same time; and will lose all its momentum. And when bodies are thrown perpendicularly upwards, the heights of their ascents are as the squares of their velocities, or as the squares of the times of their ascending.

N. B. Throughout this chapter no notice has been taken of the resistance, which the air offers to the motion of bodies.

CHAPTER VI.

THE METHOD OF ASCERTAINING THE SITUATION OF THE CENTRE OF GRAVITY, AND AN ENUMERATION OF ITS PRINCIPAL PROPERTIES.

THE definition and the nature of the centre of gravity having been shewn in the preceding pages, we shall in the present chapter shew the method of finding its situation in a system of bodies, as well as in a single body or figure; after which we shall state its various properties, the knowledge of which is of the utmost importance in the study of natural philosophy, and especially in mechanics.

When the common centre of gravity of two bodies is to be determined, their quantities of matter and distance from each other being known;

draw a straight line from the centre of gravity of one of the bodies, as A, fig. 10. Plate I. to the centre of gravity of the other, B. (which centres we shall, for the present, suppose to be known; for it will presently be shewn how to find the centre of gravity of a single body.) Then divide this line in E, so that its parts BE, AE may be to each other in the proportion of A to B, and E is the centre sought. For example, let A weigh 3 pounds, B 2 pounds, and let the distance AB be 20 feet. Say as 3 is to 2, so is BE to AE; then by composition (Eucl. p. 18. B. V.) say as 3 *plus* 2, viz. 5, is to 2 so is BE *plus* AE, viz. 20 feet, to a fourth proportional, which, by the common rule of three, is found to be 8, and is equal to AE; so that the centre of gravity E is 8 feet distant from A.

When the centre of gravity of three bodies, as A, B, and D, fig. 11. Plate I. is to be determined, their quantities of matter and distances being known; you must in the first place find the centre of gravity E between any two of those bodies, as of A and B, after the manner mentioned above. Then imagine that the two bodies A and B are both collected in the point E, and lastly find the centre of gravity between E and D, which will be the common centre of gravity of the three bodies; viz. draw the straight line DE, then as the sum of the matter in A and B, is to the matter in D, so is DC to CE; and, by composition, say as the sum of the matter in A, B and D, is to D, so is DE to CE; which

which gives C, the common centre of gravity of the three bodies.

In the same manner the centre of gravity of four, or more, bodies may be determined; viz. by conceiving the matter of three of those bodies to be collected in the common centre of gravity of those three bodies, and then finding the common centre of gravity of the last mentioned centre and the fourth body, &c.

The centre of gravity of a single body may be easily determined by the following general method, viz. by supposing the body to be divided into two or more parts, and then finding the common centre of gravity of those parts, which will be the centre of gravity of the body itself. But in certain regular figures, such as a circular surface, a sphere, a cube, &c. it is evident that the centre of the figure must coincide with the centre of gravity; for if in the circle, for instance, fig. 12. Plate I. you divide the area into two equal parts ADB and AEB, it is evident that the common centre of gravity of those two equal parts must be somewhere in the line AB; and if, by cutting the circle in any other direction ED, you divide the area into two other equal parts EAD, and DBE, it is evident that the common centre of gravity of those two parts must be somewhere in the diameter ED; therefore the centre of gravity of the circle must be in the intersection of the two diameters AB, ED; viz. at C, which is the centre of the circle.

In a right lined plane triangle, as ABD, fig. 13. Plate I. the centre of gravity may be easily found by dividing any two of its sides, as AD and BD, each into two equal parts at F and E, and by drawing straight lines from those points of division to the opposite angles; the intersection C of those two lines being the centre of gravity of the triangle; for since the line AE divides the triangle into two equal parts, (Eucl. p. 1. B. VI.) the common centre of gravity of those two parts must be somewhere in the line AE, and for the same reason the common centre of gravity of the two parts ABF, and FBD must be somewhere in the line BF; therefore the centre of gravity of the whole triangle must be at C, the intersection of the two lines AE, BF.

If the figure be terminated by more than three straight lines, as ABCDE, fig. 14. Plate I. its centre of gravity may be found by dividing it into any convenient number of triangles, as ABC, BCF, DFE, then by finding the centre of gravity of each triangle, and lastly by finding the common centre of gravity of all the triangles, which is to be done in the same manner as the centre of gravity between three or more bodies was determined above. In a similar manner the centre of gravity of irregular solids may frequently be found.

There are however several figures in which the centre of gravity cannot be easily found by the above described methods, at least not with great accuracy. But a more general, and accurate method

method of finding the centre of gravity, is derived from the doctrine of fluxions, which will be found explained in the note (1).

The

(1) Imagine that at D, E, F, G, H, fig. 15. of Plate I. there are so many weights affixed to the inflexible line AB; and let C be the centre of gravity of the said line and weights together; so that when the point C rests upon a fulcrum, neither end will preponderate, and of course the whole loaded line will remain perfectly balanced.

It has been shewn in the preceding pages that the momentum or force of any weight, as H, (viz. the body suspended at H) to raise the opposite end A of the line, or lever, is expressed by the product of its quantity of matter multiplied by its distance from the fulcrum, or centre of gravity C; viz. by $H \times HC$ (for by the letters D, E, F, &c. we express the weights of the respective bodies). Therefore, since the line with all the weights is perfectly balanced, it follows that the sum of the momenta of all the weights which lie on one side of C, must be equal to the sum of all the momenta, which lie on the other side of C; viz. $H \times HC + G \times GC + F \times FC = D \times DC + E \times EC$; that is $H \times \overline{BC - BH} + G \times \overline{BC - BG} + F \times \overline{BC - BF} = E \times \overline{EB - BC} + D \times \overline{DB - BC}$; or $H \times BC - H \times BH + G \times BC - G \times BG + F \times BC - F \times BF = E \times EB - E \times BC + D \times DB - D \times BC$. Then by transposition we have $H \times BC + G \times BC + F \times BC + E \times BC + D \times BC = H \times BH + G \times BG + F \times BF + E \times EB + D \times DB$; which equation by division becomes $BC = \frac{H \times BH + G \times BG + F \times BF + E \times EB + D \times DB}{H + G + F + E + D}$ which

shews that the distance of the centre of gravity, C, from
the

The principal properties of the centre of gravity are as follows :

I. *If two bodies be connected together by means of an*

the extremity B, is equal to the quotient of the sum of the products of all the weights multiplied each by its distance from B, divided by the sum of all the weights.

Hence we derive the following general rule for finding the centre of gravity in a system of bodies ; viz. *Assume a point at one extremity of the system ; multiply the weight of each body by its distance from that point ; divide the sum of the products by the sum of the weights, and the quotient will express the distance of the centre of gravity from the assumed point.*

Take notice that if the above-mentioned point be assumed not at the extremity of the system, but any where between the bodies, as between D and H, fig. 15. plate I ; then the products by their respective distances, of the bodies on one side of the assumed point, must be considered as negative, whilst the other are considered as positive : and they must be all added together agreeably to the common algebraical rule for adding positive and negative quantities together. The result then, according as it turns out positive or negative, will shew the distance of the centre of gravity from the assumed point, either on the positive or on the negative side of that point.

The rule may be applied to the cases of single bodies, as for instance, for finding the centre of gravity of a triangle, of a cone, of a sphere, &c. by imagining the said figure to be resolved into an infinite number of parts ; for by the method of fluxions, the sum of the momenta, as also of the weights of all those parts, may be easily ascertained.

Thus

an inflexible line or rod, as A and B, fig. 22. Plate I. and the line or rod be supported by a prop, or (as it is commonly

Thus let the figure be a plane, as A D C, fig. 16. Plate I. whose axis is A B, and whose parts are supposed to be endued with gravity. Imagine this figure to be resolved into an infinite number of weights F G, fg , &c. all perpendicular, or all alike inclined, to the axis A B, and let x represent the distance A E of the little weight F G from A. Then the breadth of one of those weights is denoted by \dot{x} (the fluxion of the axis A E); therefore one of those infinitely small weights is expressed by $FG \times \dot{x}$; the fluent of which, when x becomes equal to the whole axis A B, is the sum of all the weights. Farther, if one of those weights be multiplied by its distance from A, the product, $FG \times \dot{x} \times x$, will express its momentum; and the fluent of this expression, when x becomes equal to the whole axis A B, is the sum of all the momenta. Therefore, agreeably to the general rule, if the fluent of $FG \times x \dot{x}$ be divided by the fluent of $FG \times \dot{x}$, the quotient will express the distance of the centre of gravity from A on the axis A B.

If the figure be a solid, imagine it to be divided into an infinite number of sections, or small weights, all perpendicular, or all alike inclined, to the axis; put the expression which denotes one of those sections, instead of F G in the above fluxional expressions; then proceed as above directed. Or, which is the same thing, call one of those sections S; then divide the fluent of $S \times x \dot{x}$, by the fluent of $S \dot{x}$, and the quotient will express the distance of the centre of gravity.

The only practical difficulty consists in finding the value of F G; or of the above mentioned section S in a solid figure, which

commonly called) *a fulcrum, placed under the centre of gravity C; the bodies will remain motionless.*

In

which value must, by means of the equation of the figure, be expressed in terms wherein x is the only variable letter. But the following examples will point out the mode of proceeding.

Example I. To find the centre of gravity of a straight line, or very slender cylinder, AB, fig. 17. plate I. whose parts may be supposed to be endued with gravity.

Let its length AB be called e , and supposing the line to be divided into an infinite number of little parts or weights. One of those parts is denoted simply by x ; (for the breadth FG is nothing) and the fluent of x is \dot{x} . Also the momentum of one of those particles is expressed by $x\dot{x}$, whose fluent is $\frac{1}{2}x^2$. Therefore, dividing the latter fluent by the former, we have $\frac{\frac{1}{2}x^2}{x}$, which when $x=e$, becomes $\frac{\frac{1}{2}e^2}{e}$ or $\frac{e}{2}$; and shews that the centre of gravity is at C in the middle of the line; viz. its distance from either extremity is equal to half the length of the line.

Example II. To find the centre of gravity in a triangle ABC, fig. 18. Pl. I. where the axis AD=a, base BC=b; EF is parallel to the base, and =y; and let AO be called x.

From the similarity of the triangles we have AD : BC :: AO : EF; viz. $a : b :: x : \frac{bx}{a} = y$. Therefore the infinitely small weight EF is denoted by $y\dot{x}$, or by $\frac{bx\dot{x}}{a}$, whose fluent is $\frac{bx^2}{2a}$; which (when x is equal the whole axis AD) becomes $\frac{ba^2}{2a}$ or $\frac{ba}{2}$. Also the momentum of the little weight $\frac{bx\dot{x}}{a}$

In this situation no one of the bodies will have more tendency towards the earth than the other ;
for

is denoted by $\frac{bx^2x}{a}$; the fluent of which is $\frac{bx^3}{3a}$, or (when x is equal AD) $\frac{ba^2}{3}$. Then $\frac{ba^2}{3}$ divided by $\frac{ba}{2}$, quotes $\frac{2a}{3}$, which is the distance of the centre of gravity from the vertex A; viz. $\frac{2}{3}$ of the axis AD.

It needs hardly be mentioned that the centre of gravity must necessarily be in the axis.

Example III. *To find the centre of gravity in a parabolic figure ABC, fig. 19. Plate I.*

Put the axis AD = a ; absciss AF = x ; and ordinate EF = y . By conics we know that the square of the ordinate EF is equal to the product of the parameter multiplied by the absciss, viz. $y^2 = px$, hence $y = p^{\frac{1}{2}}x^{\frac{1}{2}}$, and EG = $2y = 2p^{\frac{1}{2}}x^{\frac{1}{2}}$; which being multiplied by x , viz. $2p^{\frac{1}{2}}x^{\frac{1}{2}}x$, represents one of the infinitely small weights, into which the parabola is supposed to be resolved; and its fluent $2p^{\frac{1}{2}}x^{\frac{3}{2}}$, or $\frac{4}{3}p^{\frac{1}{2}}x^{\frac{3}{2}}$ represents the sum of all the weights.

Farther $2p^{\frac{1}{2}}x^{\frac{1}{2}}x\dot{x}$; or $2p^{\frac{1}{2}}x^{\frac{3}{2}}\dot{x}$ is the momentum of the little weight; whose fluent, which represents the sum of all the momenta, is $2p^{\frac{1}{2}}x^{\frac{5}{2}}$, or $\frac{4}{5}p^{\frac{1}{2}}x^{\frac{5}{2}}$. Then dividing the

latter fluent by the former, we have $\frac{4}{5}p^{\frac{1}{2}}x^{\frac{5}{2}} \div \frac{4}{3}p^{\frac{1}{2}}x^{\frac{3}{2}} = \frac{3}{5}x$, or (when x is equal to the whole axis AD) = $\frac{3}{5}a$; so that the centre of gravity is distant from the vertex A, $\frac{3}{5}$ of the whole axis AD.

Example

for since C is their common centre of gravity, the distances AC and CB, are inverfely as the weights of

Example IV. *To find the centre of gravity of a right cone ABC, fig. 20. Plate I.*

Put the axis or altitude $AD=a$, diameter of the base $BC=b$; abscifs $AO=x$ and ordinate $EO=y$.

Then from the fimilarity of the triangles AOE, ADB, we have $AD : DB :: AO : EO$; viz. $a : \frac{b}{2} :: x : y = \frac{bx}{2a}$, and, putting c for the circumference of a circle whose diameter is unity, the circumference of a circle whose diameter is EF, or $2y$, will be $2cy$; whose area is $2cy \times \frac{1}{2}y$; viz. cy^2 , or (by fubftituting $\frac{bx}{2a}$ for its equal y) $\frac{cb^2x^2}{4a^2}$. Therefore

$\frac{cb^2x^2}{4a^2}x$ represents one of the infinitely fmall weights into which the cone is fupposed to be refolved, and its fluent, $\frac{cb^2x^3}{12a^2}$, is the fum of all thofe weights. Alfo $\frac{cb^2x^2}{4a^2}x\dot{x}$ is

the momentum of the little weight, and its fluent, $\frac{cb^2x^4}{16a^2}$, is the fum of all the momenta. Then dividing the latter fluent by the former we have $\frac{cb^2x^4}{16a^2} \div \frac{cb^2x^3}{12a^2} = \frac{3}{4}x$; or (when x becomes equal to the whole axis AD) $\frac{3}{4}a$; which fhews that the diftance of the centre of gravity from A on the axis AD is equal to $\frac{3}{4}$ of the whole axis AD.

Example V. *To find the centre of gravity of an hemisphere ABO, fig. 21. Plate I.*

Put the axis or radius $AD=a$; $DP=x$; and MP, which is parallel to the bafe, $=y$. Then PMD being a right angled triangle, we have $\overline{MP}^2 = \overline{MD}^2 - \overline{DP}^2$; viz.

of those bodies; viz. AC is to CB as the body B is to the body A. Now should the rod be moved from its situation AB into the situation FE, the body B would describe the arch BE, and the body A would describe the arch AF, which arches represent the velocities of those bodies; for they are the spaces through which they move in

$yy = a^2 - x^2$. And, putting c for the circumference of a circle whose diameter is unity, the circumference of a circle whose diameter is ME, or $2y$, will be $2cy$, and its area will be $2cy \times \frac{1}{2}y$; viz. cy^2 , or (by substituting for yy its value as found above, viz. $a^2 - x^2$) $ca^2 - cx^2$; and this is a section of the hemisphere parallel to the base. Then $\overline{ca^2 - cx^2} \times \dot{x}$ is one of the infinitely small weights into which the hemisphere is supposed to be divided; and its fluent $\overline{ca^2x - \frac{cx^3}{3}}$ is the sum of all those weights. Also $\overline{ca^2 - cx^2} \times x\dot{x}$ is the fluxion of the momentum of the small weight; the fluent of which, viz. $\frac{ca^2x^2}{2} - \frac{cx^4}{4}$, is the sum of all the momenta. And, when x is equal to the whole axis AD, those two fluents become $\left(ca^3 - \frac{ca^3}{3}, \text{ or } \frac{3ca^3 - ca^3}{3}, \text{ or } \frac{2ca^3}{3}\right)$; and $\left(\frac{ca^4}{2} - \frac{ca^4}{4}; \text{ or } \frac{2ca^4 - ca^4}{4}, \text{ or } \frac{ca^4}{4}\right)$. Then, dividing the latter fluent by the former, we have $\frac{ca^4}{4} \div \frac{2ca^3}{3} = \frac{3ca^4}{8ca^3} = \frac{3}{8}a$; so that the centre of gravity is distant from the point D, $\frac{3}{8}$ of the axis, or of the radius, AD.

the same time. But it has been demonstrated by the geometricians, that those arches bear the same proportion to each other as the radii or distances CB , CA ; viz. BE is to AF as BC is to AC ; and it has been shewn above, that BC is to AC as the body A is to the body B ; therefore it follows, that the arch BE is to the arch AF as the body A is to the body B . But of four proportional quantities, the product of the extremes is equal to the product of the means; therefore the product of the body B , multiplied by the arch BE (which constitutes the momentum of B) is equal to the product of the body A multiplied by the arch AF (which constitutes the momentum of A); so that their momentums being equal, those bodies will balance each other, and of course they will remain at rest. It is evident that the same reasoning is applicable to the common centre of gravity of any number of bodies, as also to the centre of gravity of a single body; viz. that if a system of bodies, that are connected together, or a single body, be placed with the centre of gravity on a fulcrum, that system, or single body, will remain perfectly balanced thereon, and as steady as if it were placed upon a flat horizontal surface.

II. *The state, whether of rest or motion, of the common centre of gravity of two bodies, will not be altered by the mutual action of those bodies upon each other.*

For, in the first place, suppose that the centre of

gravity is at rest ; and that the two bodies, in consequence of their mutual attraction, approach each other in a certain time ; it follows from the foregoing theory, that the spaces through which they move must be inversely as their weights, and of course, the remaining spaces between their last situations and the centre of gravity, will remain in the same proportion to each other as the original distances ; therefore the centre of gravity will not be moved from its original place.—An example will render this explanation more evident.

Let the body A, fig. 23, Plate I. weigh 2 pounds, and B, 6 pounds. The distances of those bodies from the centre of gravity C, are inversely as the weights of those bodies ; viz. BC, is 10 feet, and AC, 30 feet ; (that is, as one to three) because the weights are as three to one. Now suppose that in consequence of their mutual attraction, those bodies begin to move towards each other ; and if in one minute B comes to the place D, having passed over the distance BD, equal to one foot ; the body A must in the same time have passed over 3 feet, and must have arrived at E ; then taking away BD from BC ; viz. one foot from 10 feet, there remains DC, equal to 9 feet ; also taking away AE from AC ; viz. 3 feet from 30, there remains CE, equal to 27 feet. Now those two remaining distances ; viz. 9, and 27, are the one to the other as one to three ; therefore, &c.

In the second place, if the two bodies, together
with

with their common centre of gravity, be in motion, the peculiar motion, which arises from their mutual attraction, cannot alter the course of their common centre of gravity ; because, as has been said above, they must move towards the centre C in the inverse ratio of their weights.

III. The same reasoning may be evidently applied to the case of three, or four, or, in short, of any number of bodies ; as also to the case of repulsion ; changing, in the latter case only, the name of *centre of attraction*, for *centre of repulsion* ; and reckoning the motion of the bodies not towards, but from that centre. Hence it may in general be concluded, that *the state of the common centre of gravity of different bodies will not be altered by the mutual action of those bodies upon each other.*

IV. *If any two bodies be carried towards the same parts with equal or unequal velocities, the sum of the momentums of both the bodies will be equal to the momentum that would arise if both the bodies moved with the velocity of their common centre of gravity.*

When the velocities of the bodies A and B are equal, the proposition is self-evident ; for then their common centre of gravity must move with the like velocity. But whether the velocities be equal or unequal, the proposition may be easily demonstrated (2).

V. *If*

(2) Let A and B, fig. 1. Plate II. be two bodies moving towards D. Let C be their common centre of gravity,

V. If two bodies be carried towards contrary parts, the difference of their momentums towards those contrary parts

gravity, and whilst A moves as far as a , let B move as far as b , and C as far as c . Since the spaces Aa , Bb , and Cc , express the respective velocities of the two bodies, and of their centre of gravity, all we have to prove is, that the sum of the products of A multiplied by Aa , and of B multiplied by Bb , is equal to the product of A plus B, multiplied by Cc ; viz. that $A \times Aa + B \times Bb = \overline{A+B} \times Cc$.

Since C is the centre of gravity, A is to B as BC is to AC, and as bc is to ac . Then alternately $BC : bc :: AC : ac$, and conversely $BC : BC - bc :: AC : AC - ac$. But $BC - bc$ is equal to $Cc - Bb$; and $AC - ac$ is equal to $Aa - Cc$; therefore $BC : AC :: Cc - Bb : Aa - Cc$. But it has been shewn above, that $A : B :: BC : AC$; therefore $A : B :: Cc - Bb : Aa - Cc$. Now, since of four proportional quantities, the product of the extremes is equal to the product of the means, we have $A \times Aa - A \times Cc = B \times Cc - B \times Bb$, and by transposition $A \times Aa + B \times Bb = \overline{A+B} \times Cc$.

When the bodies do not move in the same straight line, the demonstration is the same; excepting only that in this case the velocity is to be reckoned not upon the path which is actually described by the bodies, but upon the path of their common centre of gravity. Suppose, for instance, that the bodies A and B, fig. 2. Plate II. move towards D, whilst their common centre of gravity C moves in the line CD; also that in a certain time those bodies have moved as far as the places a and b respectively, at the same time that their common centre of gravity has moved from C to c .

From

parts (which is equivalent to the sum of their momentums towards the same part) will be equal to the momentum that would arise if both the bodies were carried towards the same part with the velocity of their common centre of gravity (3).

VI. What

From A, B, *a*, and *b*, drop BH, AF, *ag* and *bK*, all perpendicular to HD, the direction of the centre of gravity. Then F*g* will represent the velocity of A, and HK, the velocity of B; for those are the real advances the bodies have made towards D. Now from the property of the centre of gravity we have $A : B :: BC : AC ::$ (since the right angled triangles ACF, BCH, are equiangular, and consequently similar, by Eucl. p. 15, B. I. and p. 4. B. VI.) $HC : CF :: Kc : cg$. Then the demonstration proceeds in the same manner as for the preceding case.

(3) Suppose that the body A, fig. 3. Plate II. moves from A to *a*, at the same time that the body B moves in a contrary direction from B to *b*, whilst their common centre of gravity moves from C to *c*. Then their respective velocities are represented by A*a*, B*b*, and C*c*. Now, in order to demonstrate the proposition, we must prove that A multiplied by A*a*, minus B, multiplied by B*b*, is equal to A multiplied by C*c*, plus B multiplied by C*c*; or,

$$A \times Aa - B \times Bb = \overline{A + B} \times Cc.$$

From the nature of the centre of gravity, we have $A : B :: BC : AC :: bc : ac$; hence alternately $BC : bc :: AC : ac$, and conversely $BC : BC - bc :: AC : AC - ac$. But it has been shewn that $A : B :: BC : AC$; therefore by substitution and alternation, we have

$$A : B :: BC - bc : AC - ac.$$

Farther,

VI. What has been said in the preceding paragraphs with respect to the centre of gravity of two bodies, may be applied to the centre of gravity of three, or four, or, in short, of any number of bodies; for it follows from the preceding propositions, that two or more of the bodies may be conceived to be concentrated into their common centre of gravity; hence the case may always be reduced to that of two bodies only.

If it be asked why, in the computation of the centre of gravity, we took no notice of the decrease of the attractive force according to the squares of the distances; the answer is, that in that case the distance being one and the same; viz. (the distance of the body A from the body B, Fig. 22. Plate I. is the same as the distance of B from A,) the computation is not altered by it*.

Farther, $BC - bc$ is equal to $Bb + Cc$, and $AC - ac$ is equal to $Aa - Cc$; therefore $A : B :: Bb + Cc : Aa - Cc$; of which four proportional quantities the product of the extremes must be equal to the product of the means; viz. $A \times Aa - A \times Cc = B \times Bb + B \times Cc$; and, by transposition, $A \times Aa - B \times Bb = \overline{A+B} \times Cc$.

* For instance, if we say that the attraction of A towards B is as the weight divided by the square of the distance; viz.

$\frac{A}{\overline{AB}^2}$, and that the attraction of B towards A is as the weight of B divided by the square of the distance; viz. $\frac{B}{\overline{AB}^2}$; those two fractions have the same denomina-

CHAPTER VII.

THE THEORY OF PERCUTIENT BODIES, THAT MOVE WITH EQUABLE MOTION.

THOUGH the present part of this work treats expressly of such properties as belong to all bodies, without noticing the particular qualities which distinguish one body from another; yet in this chapter it will be necessary to take notice of one peculiarity only; namely, of the different effects which, in the collision of bodies, are produced by their being *elastic* or *non-elastic*; the meaning of which words will be explained in the following definitions.

1. A body *perfectly hard* is that whose figure is not in the least altered by the stroke, or collision, of another body.

2. A body *perfectly soft* is that whose figure is altered by the least impression, and which is destitute of the power of recovering its original figure.

3. An *elastic body* is that which yields to the impression of another body, but afterwards recovers its figure. And,

tor, $\frac{AB}{B}^2$; consequently, to say that $AC : BC ::$
 $\frac{AE}{A}^2 : \frac{AB}{B}^2$ is the same thing as to say, that $AC :$
 $BC :: B : A$.

4. It

4. It is called *perfectly elastic* when it recovers its original figure entirely, and with the same force with which it lost it; otherwise it is called *imperfectly elastic*.

5. One body is said to strike *directly* on another body, when the right line, in which it moves, passes through the centre of gravity of the other body, and is perpendicular to the surface of that other body.

Though there are innumerable gradations from a body perfectly hard, to one perfectly soft; or between the latter and a body perfectly elastic; yet we cannot say with certainty that a body perfectly possessed of any of the above mentioned qualities does actually exist. It is however certain that our endeavours have not been able to deprive certain bodies of the least degree of their elasticity, by mechanical means.

The object of the theory of percussive bodies is to determine the momentums, the velocities, and the directions of bodies after their meeting; which we shall lay down, and explain, in the following propositions. But it must be observed, that throughout this chapter we only speak of bodies which move with equable motion, that is, of such as describe equal spaces in equal portions of time; and we do likewise suppose that the bodies move in a non-resisting medium, and that they are not influenced by any other action, excepting the single impulse, which puts them in motion: for though
such

such simple and regular movements never take place in nature; yet when their theory is once established, the complicated cases, wherein the resistances of mediums and other interfering causes, are comprehended, may be more commodiously examined; and proper allowances may be made agreeably to the nature of those causes.

I. If bodies moving in the same straight line, strike against each other, the state of their common centre of gravity will not thereby be altered; viz. it will either remain at rest, or it will continue to move in the same straight line, exactly as it did before the meeting of the bodies.

This proposition is so evidently deduced from the properties of the centre of gravity, as mentioned at N° II and III. in the preceding chapter, that nothing more needs be said about it in this place.

II. Let there be two non-elastic bodies ; and if one of them move in a straight line, whilst the other is at rest in that line, or is moving in the same direction, but at a slower rate, or is moving in the contrary direction ; viz. towards the body first mentioned ; then those bodies must necessarily meet or strike directly against each other, and after the stroke they will either remain at rest, or they will move on together, conjointly with their common centre of gravity.—Their momentum after the stroke will be equal to the sum of their momentums before the stroke, if they both moved in the same direction, but it will be equal to the difference of their momentums if they moved

moved in contrary directions.—Their velocity after the stroke will be equal to the quotient that arises from dividing the sum of their momentums, if they both moved the same way, or the difference of their momentums, if they moved in contrary directions, by the sum of their quantities of matter.

That in any of the above mentioned cases the two bodies must meet, and strike against each other, is so very evident as not to require any farther illustration.

That after the stroke those two bodies must either remain at rest, or they must move together, conjointly with their common centre of gravity, is likewise evident; for as the bodies are not elastic, there exists no power that can occasion their separation.

With respect to the momentum, it may be observed, that when the two bodies meet, whatever portion of momentum is lost by one of them must be acquired by the other; since, according to the third law of motion, action and re-action are always equal and contrary to each other; therefore, if before the stroke the bodies moved the same way, their joint momentum after the stroke will be equal to the sum of their momentums before the stroke. If one of the bodies was at rest, then, as its momentum is equal to nothing, the joint momentum will be equal to the momentum of the other body before the meeting. If the bodies moved towards each other, then their momentum after the meet-

ing will be equal to the difference of their former momentums; and if in this case their momentums are equal, then their difference vanishes; hence the bodies will remain motionless after their meeting.

The last part of the proposition is likewise evident; since it has already been shewn, that in equable motion, the velocity is equal to the quotient of the momentum divided by the quantity of matter.

When the weights and velocities of the two bodies before their meeting are known, their velocity after the meeting may be determined by the following general method.

Let A and B, in fig. 4, 5, 6, and 7, of Plate II. which represent the above mentioned cases, be the two bodies; let C be their common centre of gravity, and D the place of their meeting. Make DE equal to DC; so that the point D may be between C and E; then DE will represent the velocity of the two bodies after their meeting; for, since the bodies after the concurrence move together conjointly with their common centre of gravity; and since it has been proved in the preceding proposition, and at N° II. of the preceding chapter, that the state of the common centre of gravity of the two bodies is not altered by their mutual action upon each other; therefore the velocity of their common centre of gravity after their meeting, must be equal to its velocity before the meeting; viz. DE must be equal to CD, and is the same as the velocity

velocity of the two bodies after their meeting, because then they move together with their common centre of gravity.

Of the above mentioned figures, it may be easily perceived, that the 4th shews when both the bodies move the same way; the 5th represents the case in which B is at rest before the stroke, and of course the two points B and D coincide; the 6th shews when the two bodies move towards each other; and the 7th shews when the two bodies move towards each other with equal momentums, in which case, after their meeting, they will remain at rest. The respective velocities of those two bodies are represented in all the four figures, by AD and BD; for they run over those distances in the same time; and AB is the difference of those velocities. Also their respective momentums are represented by the product of the weight of A multiplied by AD, and the product of the weight of B multiplied by BD. The momentum of both the bodies together after their meeting, is represented by the product to their joint weight multiplied by DE (1).

Since

(1) The following is an example of the numerical computation of the first case, fig. 4, which will be sufficient to indicate the manner of calculating the other cases.

Let A weigh 10 pounds, and move at the rate of 4 feet per minute.

Let

Since when one of the bodies is at rest, the velocity after the meeting is equal to the quotient of the velocity of the moving body, divided by the sum of the quantities of matter of both the bodies; it follows that the larger the body at rest is, the smaller will the velocity be after the meeting. For

Let B weigh 6 pounds, and move at the rate of 2 feet per minute.

And let the distance A B be 32 feet.

The centre of gravity is found by saying $16 : 32 :: 10 :$

$BC = \frac{32 \times 10}{16} = 20$ feet; hence $AC = 12$ feet. (See page 75.)

Put $BD = x$, and AD will be equal to $32 + x$. Then the time employed by A in moving from A to D, is equal to the quotient of the space $32 + x$, divided by its velocity; viz. it is $\frac{32 + x}{4}$. And the time employed by B in moving from B to D, is equal to the quotient of the space x , divided by the velocity of B; viz. it is $\frac{x}{2}$. But since the bodies meet at D, those times must be equal; that is,

$\frac{32 + x}{4} = \frac{x}{2}$; hence $64 + 2x = 4x$; and $x = 32 = BD$.

Then $DE = DB + BC = 32 + 20 = 52$ feet; that is, after the meeting, the two bodies will move from D to E; (viz. over 52 feet) in as much time as each of them employed in going to D; that is, 16 minutes. Therefore, to find how many feet per minute the bodies will run over after the meeting, divide 52 by 16, and the quotient $3\frac{1}{4}$ shews that they will move at the rate of $3\frac{1}{4}$ feet per minute.

instance, if the moving body A weigh one pound, and move at the rate of one foot per minute, whilst the body B at rest weigh one pound also, the velocity after the concurrence will be half a foot per minute; half a foot being the quotient of one foot, divided by the sum of their quantities of matter; viz. 2 pounds. If *cæteris paribus* B weigh 10 pounds, then the velocity after the concurrence will be the 11th part of a foot per minute. If B weigh 100000 pounds, then the velocity after the concurrence will be the 100001th part of a foot per minute; and in short, when B is infinitely bigger than A, the velocity after the concurrence will be infinitely small, which is the same thing as to say, that in that case, after the stroke, the bodies will remain at rest. And such is the case when a non-elastic body strikes against an immoveable obstacle.

III. *If a body in motion strikes directly against another body, the magnitude of the stroke is proportional to the momentum lost, at the concurrence, by the more powerful body.*

According to the third law of motion, action and re-action are equal and contrary to each other; therefore whatever momentum is lost by one of the bodies, is acquired by the other. Or the magnitude of this acquired momentum (which is the effect of the stroke) is as the momentum lost by the more powerful body; it being by the quantity of

of the effect that we measure the quantity of the action.

IV. *When a given body strikes directly against another given body, if the latter be at rest, the quantity of the stroke is proportional to the velocity of the former body.—If the second body be moving in the same direction with the first, but at a slower rate, the magnitude of the stroke will be the same as if the second body stood still, and the first impinged upon it with a velocity equal to the difference of their velocities.—And lastly, if the bodies move directly towards each other, the magnitude of the stroke is the same as if one of the bodies stood at rest, and the other struck it with the sum of their velocities.*

The momentum of a given body is proportionate to its velocity; for with a double velocity the momentum is double, with a treble velocity the momentum is treble, and so on; therefore, as long as the body remains the same, the magnitude of the stroke, being proportional to the momentum, must likewise be proportional to the velocity. And when one of the bodies is at rest, the magnitude of the stroke is evidently proportional to the velocity of the moving body.

V. *It follows from the foregoing theory, that the mutual actions of bodies, which are inclosed in a certain space, are exactly the same, whether that space be at rest or move on uniformly and directly.*

For if the motion of the space adds to the velocity of those bodies within it, which move the same

way, it takes away an equal portion of velocity from those bodies within it which move the contrary way ; thus, all the motions of the bodies in a ship are performed in the same manner, and the same effects are produced on each other, whether the ship be at rest or move uniformly forwards.

The attentive reader must have perceived, that in the explanation of the preceding cases the distances have been reckoned from the centres of the bodies ; whereas it is evident that the thickneses of the bodies must be deducted from those distances ; for the bodies do not strike against each other with their centres, but with their surfaces. It must be observed, however, that when the distances are very great, and the bodies proportionately very small, it is immaterial whether the distances be reckoned from the centres or from the surfaces of the bodies. But if great accuracy be wanted, the thickness of the bodies may be easily allowed for in the computation.

A similar observation may be made with respect to the shapes of the bodies ; viz. that they have been represented as being globular, for the purpose of rendering the explanation short and perspicuous.

Having hitherto treated of unelastic bodies, that is, of such bodies as are either perfectly hard or perfectly soft, it is now necessary to state the rules of congress which take place with elastic bodies.

At the commencement of the present chapter, elasticity has been said to be that property by which bodies yield or suffer their figure to be altered by the pressure of other bodies; but which, on the removal of the pressure, recover their original figure of their own accord.

This recovery of the figure is performed with greater or less quickness, and with more or less exactness, in different bodies; which differences constitute the various degrees of elasticity. And those bodies, which recover their figure completely, and as quickly as they lost it, are said to be *perfectly elastic*.

Though we are acquainted with the effects and the laws of elasticity, in a manner sufficient to render that property subservient to our purposes, yet the cause of that property is by no means understood; nor has any hypothesis been offered in explanation of it, which may be said to be sufficiently plausible.

Let a string, A B, fig. 8. Plate II. be stretched between, and be firmly fastened on, two immovable supports at A and B; and if, by applying a finger at C, this string be pulled towards D, the string will be found to resist that effort with a force greater and greater, the farther it is pulled from its original straight situation. When disengaged from the finger, the string will not only return to its original straight situation, but it will go beyond it; viz. towards E, and nearly as far from the straight

situation *ACB*, as *ADB* is from it, after which it will again bend itself towards *D*, and so on, vibrating backwards and forwards, but deviating continually less and less from the straight situation, until at last it remains at rest in its original situation *ACB*.

Now when the string is first disengaged from the finger, its elastic force draws the part *D* towards *C*; but this force decreases in proportion as the part *D* comes nearer to *C*, and when the part *D* is arrived at *C*, that is, when the string is in its straight situation, the above-mentioned force is infinitely little or is equal to nothing; but by that time the part *D*, having been impelled by a continual though decreasing force, will have acquired a momentum which carries it towards *E*; viz. beyond the straight situation; but as soon as the string goes beyond *C*, the elastic force begins to act again in a direction contrary to that of the momentum. This action becomes stronger and stronger the farther the middle part of the string goes from the straight direction, and of course it gradually diminishes the above-mentioned momentum, until at last the momentum being entirely spent, the string begins again to move towards *C*, in virtue of the elastic force, and so on.

The extent of the vibrations becomes continually shorter and shorter, on account of the resistance of the air, and of the want of perfect pliability, or of perfect elasticity, in the parts of the string; which

which two causes continually tend to diminish the momentum, &c.

If the string be moved out of its straight direction, not by the application of a finger, but by a body E, fig. 9. Plate II. falling upon it at C from the height H, it will be readily understood that the momentum of the body will easily impel the string towards D; but the re-action of the string on the body will gradually diminish the momentum of the latter, and the farther the string is carried from the straight situation, the stronger will its re-action be, until at last the body, having lost all its momentum, will be carried back again towards C by the elastic force of the string; and in its way back, the constant though decreasing action of this elasticity from D to C, will give it a momentum which will carry it up towards H; and the body would ascend precisely up to H, were it not for the above-mentioned causes of obstruction; viz. the resistance of the air, &c.

It is almost superfluous to observe that the greater the height is from which the body E falls upon the string, the farther will the string be removed from its straight situation, and of course the stronger will its re-action be.

This explanation of the elasticity of the string may be applied to all sorts of elastic bodies; for the surface of every one of them will be bent more or less by any stroke or pressure, and will afterward, recover its original form by re-acting the contrary

way with a force, which, in perfectly elastic bodies, is equal to the pressure or stroke received.

If instead of the string, whose ends are immovable, &c. we imagine that two equal bodies like A and B, fig. 10. Plate II. being impelled by equal and contrary forces, directly strike against each other at C, it is evident that the contiguous surfaces of both will be bent inwardly, and that the elastic force of A will drive the body B back from C towards B, (as the string did in the preceding case) at the same time that the elastic force of the body B will impel A back with equal force from C towards A, so that in this case the bodies after the stroke will recede from each other; whereas had they been non-elastic, they would have remained stationary at the place of their congress C.

We must now determine the effects produced after the congress of bodies that are *perfectly* elastic; from which the laws of congress amongst those that are *imperfectly* elastic may afterwards be easily deduced.

VI. *When two bodies that are perfectly elastic strike directly on each other, their relative velocity (by which is meant the excess whereby the velocity of the swifter body exceeds that of the slower) will be the same before and after the stroke; viz. they will recede from each other with the same velocity with which they approached before the stroke.*

It has been shewn above, that when two given bodies strike on one another, the magnitude of the stroke

stroke is proportional to their respective velocities. By the definition, a perfectly elastic body has been said to be that which recovers its figure with the same force with which it lost it. Therefore in perfectly elastic bodies the restoring force is equal to the compressing force: so that if the momentums of the bodies produced a certain compression, the elastic force must re-act on the bodies with the like force; hence the bodies will be forced to recede from each other with the same velocity wherewith they approached each other.

It is a natural consequence of this demonstration, *that in equal times taken before and after the stroke, the distances of the bodies from one another will be equal, and therefore in equal times their distances from their common centre of gravity will likewise be equal.*

Thus much being premised, the laws of congress in bodies that are perfectly elastic may be easily determined. But in order to comprehend the solution of the following cases, the reader should recollect and keep in view the following particulars, which have all been sufficiently proved in the preceding pages; viz. 1st, that the distances of two bodies from their common centre of gravity are inversely as their weights: 2dly, that the state of rest or of uniform motion of the centre of gravity of bodies is not altered by the mutual action of those bodies on each other: 3dly, that in bodies that are perfectly elastic, the restoring is equal to the compressing

pressing force: 4thly and lastly, that the distances of the bodies from each other, and from their common centre of gravity, are equal in equal times taken before and after the stroke.

Now let A and B, fig. 10. Plate II. be two equal bodies perfectly elastic; C is their common centre of gravity, which, since the bodies are equal, stands midway between them. Let both the bodies be impelled with equal force, directly towards each other, in consequence of which they will move in a certain time (for instance, a minute) from their respective places to C, where they meet. After the impulse their elasticity will impel each body back towards its original place, so that at the end of one minute from the time of their meeting they will be found precisely where they were a minute before their meeting, viz. at A and B.

Let A and B, fig. 11. Plate II. be two equal and perfectly elastic bodies, as in the last case, and let one of them, for instance B, be at rest, whilst the other body A moves towards it, so as to reach it in one minute's time. Here AB represents the velocity of it, and CB represents the velocity of the centre of gravity C; for in the same time that A comes from the place A to the place B, C must come from C to B, therefore at the end of one minute after the stroke the centre of gravity must be at F, viz. as far from B as its original situation C was from the place B a minute before the stroke; but a minute after the stroke the body A must

must be as far from the centre of gravity, viz. from F, as it stood a minute before the stroke; therefore it must stand at B. Now as the same reasoning may be applied to any other equal times taken before and after the stroke, such as half a minute before and half a minute after, &c. therefore in this case the body A after the stroke will remain stationary at B, and the body B will move on with the velocity that the body A had before the stroke.

Having thus explained two of the simplest cases of congress in a separate manner for the sake of perspicuity, I shall now give one general rule for the solution of all the other cases, which are delineated in Plate II. from fig. the 12th to fig. the 20th inclusively, in which figures A and B represent the two bodies; C is their common centre of gravity; D the place at which they meet. AD expresses the velocity of A; BD the velocity of B, and CD that of the centre of gravity.—Then the rule for determining the velocities after the stroke is as follows:

Take a point E in the line AB, produced if necessary, so that the distance CE be equal to CD; then after the stroke the right line EA will express the velocity of the body A from E towards A, and the right line EB will express the velocity of B from E towards B.

In any one of those cases the centre of gravity C must move from C to D, and after the stroke, from D forwards to a distance DF equal to DC in
a portion

a portion of time equal to that in which A and B employ for coming from the places A and B to D.

Make Fa equal to CA ; and since in equal times, taken before and after the stroke, the distances of the bodies from the common centre of gravity are equal, therefore when the centre of gravity is at F the body A will be at a ; so that after the stroke it will move from D towards a , and Da , which it has passed over in that time, will represent its velocity. But because CE is equal CD , or to FD , and CA , is equal to Fa , the difference of the right lines CE , CA will be equal to the difference of the right lines FD , Fa ; viz. EA will be equal to Da . But Da represents the velocity of the body A after the impulse, therefore its velocity will also be represented by EA . And since the relative velocity of the bodies before and after the stroke is the same, and EA represents the velocity of the moving body A; therefore the velocity of the body B, moving from E towards B, is of course represented by the right line EB .

For the better illustration of this theory, I shall briefly mention the meaning of the figures which exhibit the various cases of the congress of bodies that are perfectly elastic, to every one of which the preceding explanation is equally applicable.

Fig. 12. is the case when B is larger than A, (which is indicated by C, the situation of the centre of gravity) B is at rest, and A strikes against it. In this case, after the stroke, both the bodies will re-
cede

cede from the point D, with the velocities EA and E B.

Fig. 13. shews when A is larger and runs against B at rest; in which case, after the stroke, both bodies will move towards F.

Fig. 14. is the case when A is larger than B, both bodies are in motion the same way, and meet at D, &c.

Fig. 15. the same as the preceding, excepting that A is less than B.

In fig. 16. A and B meet at D, where A remains at rest.

In fig. 17. after the stroke the equal bodies A and B recede with interchanged velocities.

In fig. 18. the bodies are proportional to their velocities, in which case the points C, F, D, and E, coincide.

In fig. 19. A remains stationary at the place of congress D.

In fig. 20. though the bodies A and B meet at D between the places A and B, yet after the stroke both bodies will move towards F. (2)

After

(2) The method of making the numerical computation of those cases will be shewn by the following example, which is adapted to the case represented in fig. 14, to which the reader is requested to direct his eye; though for want of room the parts of that figure be not drawn in the due proportion.

Let

After the explanation of the preceding cases, the method of determining the velocities after the stroke, when the bodies are not perfectly elastic, may be easily understood.

Thus

Let A and B be two perfectly elastic bodies. A weighs 2 pounds, and moves at the rate of 8 feet per second. B weighs one pound, and moves the same way at the rate of 5 feet per second; and let the distance A B be 12 feet.

1. To find the centre of gravity C, we have $A + B : B :: AB : CA$, viz. $3 : 1 :: 12 : 4$; so that $AC = 4$, and $CB = AB - AC = 8$.

2. To find the distance BD, put $BD = x$; and since the distances AD and BD are run over in the same time, the former at the rate of 8, and the latter at the rate of 5 feet, per second; therefore we have $\frac{x}{5} = \frac{x + 12}{8}$; hence $8x = 5x + 60$; and $3x = 60$; or $x = 20 = BD$.

3. If the distance BD, viz. 20, be divided by the velocity of B, viz. 5, the quotient 4 is the number of seconds, during which the bodies moved from their respective places A and B, to the place of their congress D.

4. $EC = CD = CB + BD = 8 + 20 = 28$; and $EA = EC - AC = 28 - 4 = 24$; which being divided by 4 (the number of seconds found above) gives 6 for the velocity of A after the stroke, in the direction from E towards A.

Also $EB = EC + CB = 28 + 8 = 36$; which, being divided by 4 (the number of seconds, &c.) gives 9 for the velocity of B after the stroke in the direction from E towards B. So that after the stroke, the bodies A and B will both continue to move the same way, but the former

Thus let A and B, fig. 21 and 22, Plate II. be two bodies imperfectly elastic; C their common centre of gravity; D the place of their meeting. Divide AC in a , so that AC may be to aC , as the force compressing the body A is to the force whereby it restores itself. Also divide BC in b ; so that BC may be to bC as the force compressing the body B is to the force whereby it restores itself. Take CE equal to CD; and lastly the right line Ea will express the velocity of A after the stroke, in the direction from E towards a , and the right line Eb will express the velocity of B after the stroke in the direction from E towards D. —

There being perhaps no body in nature which may be said to be perfectly elastic, the rules given for determining the velocities of bodies that are perfectly elastic, cannot be verified experimentally; but the deviation of the experimental result from the rules is proportionate to what the bodies want

at the rate of 6, and the latter at the rate of 9 feet per second.

In the like manner may the other cases be calculated.— I have given this method of adapting the calculation to the figures, or of expressing the parts of the diagrams, by means of numbers, in preference to the complex rules which have been given for this purpose by certain learned writers, because the latter are seldom remembered, and are difficultly applied to the solution of the various cases of impact amongst bodies that are possessed of perfect elasticity.

of perfect elasticity ; and this deviation is taken into the account after the manner mentioned above.

The precise degree of elasticity of which any particular body is possessed, must be ascertained by means of actual experiments on the body itself, which experiments differ according to the various nature of the bodies.

When more than two bodies move in the same straight line, the determination of the velocity of each body, after the various impacts with each other, cannot be comprehended under any general rules, the variety of cases being too great, and sometimes very intricate ; yet when any particular case presents itself, the preceding rules will be found sufficient for the solution of it, viz. for ascertaining the velocities, &c. But in the solution of such cases, the operator must take care to calculate, in the first place, the velocities of those two bodies which appear from the circumstances of the case to meet first ; then to substitute those new velocities which are the real velocities of those two bodies after their meeting, and with them to calculate (according as any one of those bodies is concerned with the second stroke) the velocities after the second congress, and so forth.

Most of the foregoing cases, both of perfectly and of imperfectly elastic bodies, might be expressed in the form of *canons*, (that is, of particular rules) and by stating bodies of various weights, and moving with various velocities, the numbers

of those canons might be increased without end. The following three paragraphs contain three such canons by way of examples, which the reader will easily perceive to be nothing more than particular cases expressed in words.

When two bodies, perfectly elastic and equal, are carried towards the same part, after the meeting (if their velocities be such as to admit of their striking against each other) they will continue to move in the same direction but with interchanged velocities. But if they be carried towards contrary parts, then after the meeting, they will go back with interchanged velocities.

If any number of equal and perfectly elastic bodies lie at rest, contiguous to each other in the same straight line, and another body equal to one of them strike the first of them in the same straight line with any velocity; then after the stroke the striking body and all the rest will remain at rest, and the last body only will move on with the velocity of the striking body.—In this case the bodies act as if they were separate; viz. when A, fig. 23, Plate II. strikes directly against B, after the stroke A will remain at rest, and B would move on with the velocity that A had if the other equal body C stood not contiguous; but as C is contiguous to it, B communicates its velocity to C, and remains itself at rest; and in the like manner C communicates the same velocity to D, and D to E; which last body E will in consequence of it be forced to move towards F with the velocity that A had at first.

If every thing remain, as in the preceding case, excepting that now two bodies A and B, fig. 24, Plate II. contiguous to each other, be impelled towards C, then after the stroke, A, B and C will remain at rest, and D and E will move off towards F with the velocity that A and B had at first.

CHAPTER VIII.

OF COMPOUND MOTION; OF THE COMPOSITION
AND RESOLUTION OF FORCES; AND OF OB-
LIQUE IMPULSES.

B*Y* compound motion is understood that movement of bodies which arises from more than one impulse; for in such cases the velocity and the direction of the body put in motion, arise from the concurrence of all the impulses, and participate of them all, under certain determinate laws, which will be specified in the following propositions.

I. When a body is impelled at the same time by two forces in different directions, the body will move not in any one of them, but in a direction between those two.

Thus if a body A, fig. 25, Plate II. be impelled by two forces, viz, one which by itself would drive it in the direction AB, and another, which by itself would drive it in the direction AD; then the body

A being

A being impelled by both those forces or impulses, at the same time, will move in a direction AC, between AD and AB; for since, according to the second law of motion, the change of motion is always made in the direction of the right line in which that force is impressed, therefore the motion of the body along the line AD is altered from the direction AD, to another direction towards AB, by the other impulse, which acts in the direction AB. And for the same reason, the motion of the body along the line AB is changed for another direction towards AD by the impulse which acts in that direction. Therefore the motion arising from those two impulses must have a direction between AD and AB.—But it will be shewn in the following proposition, how much this new direction will deviate from AD, and from AB.

II. *When a body is impelled at the same time by two forces in different directions, if two lines be drawn from the place in which the body receives the double impulse, in the directions of those impulses; and the lengths of those lines be made proportionate to the impelling forces; also through the end of each of those lines a line be drawn parallel to the other, a parallelogram will thereby be formed; and if a diagonal line be drawn from the place where the body receives the double impulse to the opposite corner of the parallelogram, the length and situation of that diagonal will represent the velocity and the direction of the body's motion, arising from the double impulse.*

Thus, suppose that the body A, fig. 26, Plate II. be impelled in the direction AD, by a force which would enable it to move at the rate of 4 feet per second; also that at the same time the same body be impelled by another force in the direction AB, which would enable it to move at the rate of 3 feet per second. Make AD equal to four, and AB equal to three (for instance, inches; or you may use any other dimension to represent feet). Through D draw DC parallel to AB; and through B draw BC parallel to AD; by which means the parallelogram ABCD will be formed. Lastly, draw the diagonal AC, and AC is the direction in which the body which is impelled by the above-mentioned two impulses, will move. Also the length AC will express the velocity of the body; so that if AC be found, either by calculation or by measuring the diagram, to be 5 inches long (1); we conclude that

(1) The length and direction of AC; viz. the angles it makes with AD and AB, may be easily found by trigonometry; it being the solution of a plane triangle, in which two sides, and the angle between those two sides, are known.

The direction of the impulses being given, the angle DAB is also known; for it is the angle which the directions of the two impulses make with each other. The angle ADC is likewise known, because it is the complement of the angle DAB to two right angles. The lines AD and DC (\equiv AB) are to each other in the proportion

that the body will move at the rate of 5 feet per second, since in the dimensions of AB and AD, inches were employed for representing feet.

That the body thus impelled by the two forces must move along the line AC, is easily deduced from the second law of motion; for since the change of motion is proportionate to the moving force impressed; if from any point *c* in the diagonal AC you draw two right lines, viz. *dc* parallel to AB, and *bc* parallel AD, those two lines will represent the deviations of the body's motion from the directions AD and AB; since by law the 2d, the change of motion is made in the direction of the moving force impressed. And those two lines are proportional to the impelling forces,

portion of the two impulses, and may be represented by any dimensions, as inches, feet, &c. Therefore in the triangle ADC the sides AD, DC, and the included angle D, are known. Hence by trigonometry we have $AD + DC : AD - DC :: \text{tangent } \frac{DAC + DCA}{2} :: \text{tangent } \frac{DCA - DAC}{2}$; whence we obtain half the sum of the

angles at the base, viz. of the angles DCA and DAC. Then half the sum, plus half the difference, is equal to the greater of those angles; viz. DCA; and half the sum, minus half the difference, is equal to the other angle DAC, which gives the direction of AC: thus all the angles will be known. Lastly, say, as the sine of the angle DCA is to the sine of the angle ADC, so is AD to a fourth proportional, which is equal to AC. Hence both the direction and the length of AC will be known.

or to the lines AB and AD , which represent those forces; viz. dc is to bc , as AB is to AD ; because (by Eucl. p. 24. B. vi.) the parallelogram $Adcb$ is similar to the parallelogram $ABDC$.

If it be said that the body thus impelled will at any time be found at any other place o out of the diagonal AC , draw om parallel to AD , and od parallel to AB ; then om and od , which represent the deviations, &c. ought to be proportional to the forces which occasion those deviations, viz. om ought to bear the same proportion to od as AD does to AB . But this is not the case, because the parallelogram $Admo$ is not similar to the parallelogram $ABDC$. Therefore the body, &c. must move along the diagonal AC , and in no other direction.

III. *When a body is impelled at the same time by three forces in three different directions, the velocity and the direction of the body's motion, which arises therefrom, must be determined by first ascertaining the course which would be produced by any two of those forces, according to the preceding proposition; and then by finding the course last found, and the third force, which will be the course sought.*

Thus if a body A , fig. 27, Plate II. be impelled by three forces; viz. with a force which by itself would enable it to move in the direction AB at the rate of four feet per minute; by a second force, which by itself would enable it to move in the direction AC at the rate of three feet per minute; and lastly, by a third force, which by itself would enable it to move at the rate of five feet per minute

nute in the direction A D. Make the lengths of the right lines proportionate to the forces, viz. A B four, A C three, and A D five, inches, or feet, &c. long. Then imagine as if the body were impelled by the first and second forces only, and, by the preceding proposition, find the compound motion arising therefrom, viz. through B draw B E parallel to A C, and draw C E through C parallel to A B; and the diagonal A E will represent the direction and the velocity of the motion resulting from those two forces. Then after the same manner find the compound motion resulting from the force represented by A E, and the third force represented by A D; viz. by drawing through E and D the lines E F and D F, respectively parallel to A D and to A E, as also the diagonal A F; and this diagonal A F will represent the course of the body, viz, the velocity and direction of its motion, arising from the above-mentioned three impulses.

The demonstration of this proposition is so evident a consequence of the preceding proposition, that it will be needless to detain the reader with a repetition of almost entirely the same words.

It appears likewise, that the like reasoning may be extended to the case of four or five, or in short, of any number of impulses.

Notwithstanding the apparent multiplicity and intricacy of such cases, an obvious remark will furnish a general rule, by means of which the place of the body at any time may be easily deter-

mined in all cases; viz. whether the impulses be single, or accelerative like the force of gravity, or whether some of them be of the former, and others of the latter sort.

The observation which furnishes the rule is, that at the end of a certain time the body which is impelled by two forces, will be found precisely at the place where it would be found if the two forces acted one after the other; the time however must not be doubled. For instance, in the case of fig. 26, Plate II. the body A is impelled by two forces, viz. one in the direction AD, which alone would drive it as far as D in one second, and another force in the direction AB, which alone would drive it to B in one second. Now if you imagine that those two forces be applied one after the other, viz. that when the body is at D, the other force impels it in the direction DC parallel to AB, and as far from D as B is from A; then C is the place where the body will be driven in one second by the compound action of both the forces.

This observation is evidently applicable to the case of four or more impulses; and hence we derive the following general rule for finding the place or situation of a body after a certain time, when the body is impelled by any given number of given forces.

Rule. Imagine as if the body were impelled by the given forces, not at once, but successively one after the other, in directions parallel to their original directions,

and each in an equal portion of time; and the last situation is the place where the body will be driven in the like portions of time, by the joint action of all the forces at the same time.

Thus in the case of fig. 27, Plate II. the first force by itself would impel the body to B in one minute; the second force would by itself impel it from B to E in one minute (B E being equal and parallel to A C); and the third force alone would impel it from E to F in one minute (E F being parallel and equal to A D); therefore the joint action of all the three forces will drive the body from A to F in one minute.

If, instead of one minute, any other portion of time be made use of, the figures arising therefrom will always be similar, so that whether the figure be larger or smaller the point F will always be in the right line A F; which likewise shews that when a body is impelled by single impulses (viz. such as produce equable motion) let their number be what it may, the course of the body between its original place A and the place F, where it will be found at the end of any time, is always rectilinear; hence the right line A F represents, as we have already observed, the direction and velocity of the body's motion.

Sometimes the directions and the strength of the impulses are so circumstanced as to produce no motion on the body; in which case the forces are said to be balanced in opposite directions; and to those cases

cases the above-mentioned rule is equally applicable. Thus if a body A, fig. 28, Plate II. be impelled by a force which in one minute's time would drive it in the direction A B as far as B, and likewise by another force equal to the former, which by itself would drive it in the direction A C opposite to the direction A B, to a place C, as far from A as B is from A; it is evident that those two equal, but opposite, impulses, acting at the same time, will not produce the least motion in the body, for they destroy each other.

Likewise if a body be impelled by three powers in three different directions, and the compound course which would be produced by two of those forces be equal and opposite to the third force, the body will not be moved by those three forces.— Thus if the body A, fig. 29, Plate II. be impelled in the direction A B by a force which in one minute would enable it to go as far as B; also by a second force, which in one minute would enable it to go in the direction A C as far as C; and lastly, by a third force, which in one minute would enable it to move in the direction A E as far as E. Find the course of the body which would arise from the joint action of the two forces A B and A C, viz. by drawing B D parallel to A C, and C D parallel to A B, and joining A D. Now if this diagonal A D happens to be equal to A E and opposite to it, that is, in the same right line, then the body A will not be moved by the joint action of those three forces; for the

the two forces AB and AC are equivalent to the force represented by AD ; but this force AD is equal and opposite to the force AE ; therefore the sum of the two forces AB and AC , is likewise equal and opposite to AE ; hence the body will not be moved from its original place A .

Since $ABDC$ is a parallelogram, the line BD is equal and parallel to AC , as also the line DC to AB ; and, in case of an equilibrium or balance of the three forces, AD has been shewn to be equal to, and in the same right line with, AE , which is the same thing as to say that AD is parallel to AE . Therefore we establish the following proposition, which is of great use in mechanics.

IV. *If a body be impelled by three powers, or, (which is the same thing) it be drawn by three powers, in three different directions, and those powers balance each other so as to leave the body at rest; then those powers must have the same proportion to each other as have the right lines (AB , BD , and AD) drawn parallel to their directions, and terminated by their mutual concurrence. And vice versa, if the lines drawn parallel to the directions of the three forces, and terminated by their mutual concurrence, bear to each other the same proportion that the forces bear to each other, then the body will remain at rest (2).*

It

(2) By trigonometry, the sides of plane triangles are as the sines of their opposite angles. Therefore in the triangle

It will be hardly necessary to observe, that a balance of forces may take place amongst any number of such forces ; so that a body may remain perfectly at rest, though acted upon by four or five, or any other number of forces. In this case the forces are sometimes called *pressures* ; for in fact they only press upon the body without being able to move it out of its place.

V. *As the joint action of several impulses compel a body to perform a certain course, so whenever we observe any particular course of a body, we may imagine that course to be produced by the joint action of two or more impulses that are equivalent to that single impulse.*

Thus finding that a body *A*, fig. 26, Plate II. has moved from *A* to *C*, we may imagine either that the body has been impelled by a single force in the direction of *AC*, and proportionate to the length of *AC*, or that it has been impelled by two forces at once, viz. by one in the direction of *AD*,

angle *ABD*, fig. 29, Plate II. *AB* is to *BD* as the sine of the angle *ADB*, or *DAC*, is to the sine of the angle *DAB* ; hence any two powers will be to each other reciprocally as the sines of the angles, which the lines representing their directions, make with the direction of the third power. Farther, *AD* is to *AB* as the sine of the angle *ABD* is to the sine of *BDA*, or *DAC* ; and in like manner the power acting according to *AE* is to the power acting according to *AC*, as the sine of the angle *ACD* is to the sine of the angle *ADC*, or *BAD*.

and

and proportionate to the length of AD ; and by another force in the direction of AB or DC , and proportionate to AB or DC . Therefore, *if two sides of any triangle (as AD and DC) represent both the quantities and the directions of two forces acting from a given point, then the third side (as AC) of the triangle will represent both the quantity and the direction of a third force, which, acting from the same point, will be equivalent to the other two, and vice versa.*

Thus also in fig. 27, Plate II. finding that the body A has moved along the line AF from A to F in a certain time; we may imagine, 1st, that the body has been impelled by a single force in the direction and quantity represented by AF ; or 2dly, that it has been impelled by two forces, viz. the one represented by AD , and the other represented by AE ; or thirdly, that it has been impelled by three forces, viz. those represented by AD , AB , and AC ; or lastly, that it has been impelled by any other number of forces in any directions; provided all those forces be equivalent to the single force which is represented by AF .

This supposition of a body having been impelled by two or more forces to perform a certain course; or, on the contrary, the supposition that a body has been impelled by a single force, when the body is actually known to have been impelled by several forces, which are, however, equivalent to that single force; has been called *the composition, and resolution*

of

of forces; and is of great use in mechanics, as will be shewn in the sequel.

In the preceding pages we have laid down the laws relative to the congress, or impact, of bodies, when the bodies strike in a direction perpendicular to each other. It is now necessary to examine those cases in which the bodies strike in an oblique direction, the effects of which could not have been conveniently explained previously to the doctrine of the composition and resolution of forces, since it depends principally on that doctrine.

It has been shewn, that if a body *A*, fig. 26, Plate II. be struck by two powers at the same time, viz. by one in the direction from *A* towards *B*, and by the other in the direction from *A* towards *D*; the body will thereby be forced to describe the diagonal *AC*. Now let this motion be reversed, viz. imagine that the two powers or bodies *K* and *L*, are at rest, and that the body *A*, advancing from *C*, along the line *CA*, strikes against those two bodies at the same time; the consequence will be, that both the bodies will be moved from their places, since they are both struck; that the impulse will be divided amongst them in the proportion of the line *AB*, which is perpendicular to the body *K*; to the line *AD*, which is perpendicular to the body *L*; and lastly, that the body *L* will be impelled towards *Q*, whilst the body *K* is impelled towards *P*.

It is evident that the force of the impulse must be divided amongst the two bodies; so that the greater is the quantity of it which is communicated to the one, the smaller must be the quantity of it which is communicated to the other; also each of those quantities must be less than the whole original force of the body A; otherwise there would be an accumulation of force without any adequate cause, which is not possible.

The force is not only divided amongst the two bodies K and L, but it is divided in the proportion of the lines BA to DA, which is easily proved thus: Since any force may be resolved into two or more forces, therefore if we divide the force represented by the line AC into two forces, such as that one of them cannot possibly act upon the body K, whilst the other acts directly against it, we shall by that means determine the question. Draw AD parallel to the surface of the body K at the point of congress; from C drop CD perpendicular to AD, and through the point of congress draw AB parallel to DC, which AB being perpendicular to AD, must likewise be perpendicular to the body K at the point of congress. Thus the force AC is resolved into the two forces AD and DC, or AB; the former of which cannot have any action upon the body K, whilst the latter acts entirely upon it. For instance, imagine that instead of a body moving from C towards A, two bodies move towards that point, viz. one in the direction DA and the other in the direction BA, and it is evident that

that of those two powers, the one in the direction BA will act entirely and directly upon the body K , whilst the other in the direction DA passes by it, and of course cannot affect it.

By the like reasoning it is proved, that when the force represented by CA is resolved into two other forces, viz. AB , which is parallel to the surface of the body L at the point of congress, and BC , or its equal AD , which is perpendicular to it, the latter only will act upon the body L ; therefore the force which acts upon the body K is represented by AB , or its equal DC ; and the force which acts upon the body L is represented by the line AD , or its equal BC ; and those two forces are equivalent to the force AC .

The inclination of the direction of the stroke to the body K , or to the line AD , which is parallel to the surface of it at the point of congress, is represented by the angle DAC ; and the inclination of the stroke to the body L , or to the line AB , is represented by the angle CAB . Now (by trigonometry) when AC is made radius, DC , or its equal AB , becomes the sine of the angle of inclination DAC ; and BC , or its equal AD , becomes the sine of the angle of inclination BAC ; therefore *the effect of the oblique force CA , is to the effect that would be produced by the same force coming in a perpendicular direction, as the sine of the angle of inclination is to radius*; which is a general and useful law in the computation of oblique impulses.

In the present instance the proportion of the oblique force CA to move the body K , is to that of the same force coming in a perpendicular direction, as the sine DC is to radius AC ; and for the body L it is as the sine BC to radius AC .

If in the above-mentioned case we imagine that one of the bodies be removed, whilst the other is fixed, we shall then form the case represented by fig. 1. Plate III. in which the body A , moving in the direction AC , strikes obliquely at C on the firm obstacle BF ; where it is plain that the magnitude of the oblique stroke is to the magnitude of the same stroke if it had come in a direction perpendicular to the obstacle, as the sine of inclination, or of incidence, (viz. as the perpendicular AB) is to the radius AC .

If a body perfectly elastic as A , fig. 1. Plate III. strike obliquely at C on the firm obstacle BF , then after the stroke this body will be reflected from that obstacle in the direction CE , in such a manner as to form the angle of reflection ECF , equal to the angle of incidence ACB .

The oblique force AC being resolved into two forces, viz. DC perpendicular to the obstacle and AD parallel to it; the effect on the plain is the same as if the body had advanced towards it directly from D , and according to the laws of congress between perfectly elastic bodies, (chap. vii.) the body A after the stroke would be sent back in the direction CD . But of the two forces into

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which the original force of the body A was resolved, this body retains the one represented by AD , since this force was not concerned in striking the obstacle; therefore after the stroke the body A is actuated by two forces, viz. one represented by CD , equal to AB , equal to EF ; and the other represented by CF , equal to DA , equal to DE , hence it must move in the diagonal CE ; and since the lines CF , FE , are respectively equal to the lines CB , BA , and the angles at B and F are equal, because they are right-angles; therefore (Eucl. p. iv. B. I.) the triangle EFC is in every respect equal to the triangle ACB ; consequently the angle of reflection ECF is equal to the angle of incidence ACB .

Some writers call the angle ACD the angle of incidence, and the angle DCE the angle of reflection; viz. the angles made by the body with the perpendicular DC ; this however does not alter the proposition, for the angle ACD is likewise equal to the angle DCE ; those angles being the complements of the equal angles ACB , ECF , to two right angles.

In any case, whenever two bodies strike obliquely against each other, whether one or both be in motion, their directions, velocities, and momentums after the stroke may be easily determined from what has been explained in the last paragraphs, together with what has been delivered concerning the direct impact of elastic and non-elastic bodies in Chap.

VII. And the following example will shew the application.

Imagine that two non-elastic bodies, A and B fig. 2. Plate III. moving, the former in the direction AC, the latter in the direction BD, do meet at CD. Let the line MG be drawn through their centres and the point of contact. From the original situations of those bodies, viz. from A and B, drop AM and BN perpendicular on MG. Then the force of each body may be resolved into two forces, viz. that of A into AM, and MC; and that of B into BN and ND.

Of its two forces, A retains the force AM, whilst the force MC is exerted against the other body. Of the two forces belonging to the body B, the force BN is retained by it, whilst the force ND is exerted against the other body. Therefore the action of those bodies upon each other is exactly the same as if they moved directly one from M and the other from N; hence whether they would, after the stroke, proceed both the same way, or different ways, and at what rate, must be determined by the rules of direct impact (chap. vii.) But when their velocities have been thus determined; for instance, it be found that if the bodies had moved directly from M and N, after the stroke the body A would have moved as far as O, whilst the body B would have moved as far as G. Then it must be recollected, that, in the present case of oblique collision, the body A has retained the force

AM ; therefore after the stroke the body A is actuated by two forces, viz. one equal and parallel to AM , and another force, which is equal and parallel to CO ; in consequence of which this body must run a compound course, which is found thus: Through the centre C draw CI equal and parallel to AM ; through I draw IE equal and parallel to CO ; then the diagonal CE exhibits the velocity and the direction of the body A after the oblique concurse.

With respect to the body B , it has been said that at the concurse this body retains the force BN , and that, if the bodies had moved directly towards each other, B would, after the stroke, have moved from D to G . Therefore through D draw DH equal and parallel to BN , and through H draw HF equal and parallel to DG ; and lastly, the diagonal DF will represent the velocity and the direction of the body B after the oblique concurse.

This is the case when the bodies are perfectly hard or non-elastic. But if they be perfectly elastic, then suppose it be found by the rules for elastic bodies that, after the supposed direct concurse, the body A would have been sent back to Q in the same time that B would have been sent back to R . Then after the oblique stroke the body A will be actuated by two forces, viz. one equal and parallel to AM , and the other equal and parallel to CQ : And the body B will likewise be actuated by two forces, viz. one equal and parallel to BN , and the other

other equal and parallel to DR. Therefore in fig. 3, through Q draw QZ equal and parallel to AM, also through Z draw IZ equal and parallel to CQ; then the diagonal CZ represents the direction and velocity of the elastic body A after the oblique concurrence.

Again through R draw the line RX equal and parallel to BN, and through X draw the line XY equal and parallel to DR, then the diagonal DX will represent the velocity and direction of the elastic body B after the oblique concurrence.

In short, the cases represented in fig. 2, and fig. 3, differ only in this, namely, that the bodies are supposed to be perfectly hard in the former, but perfectly elastic in the latter.

We might now proceed to examine the particulars relative to the congress of three or more bodies, as also of bodies of different shapes, for hitherto we have supposed the bodies to be quite spherical, &c. but this we shall omit, first, because the reader may, by a little exertion of his ingenuity, easily derive it from what has been already explained; and secondly, because the particular examination of all the branches of compound motion would swell the size of the work far beyond the limits of an elementary book*.

* For further information relative to this subject, the reader may consult the 2d book of s'Gravesande's *Mat. Elem. of Nat. Phil.* edited by Desaguliers.

CHAPTER IX.

OF CURVILINEAR MOTIONS.

HITHERTO we have considered the compound motion which arises from simple impulses, or such as produce equable motion. It will now be necessary to apply the above-mentioned rules to the cases of that sort of compound motion, which arises from the joint action of a simple and of an accelerative or continue force; in which case it will be found, that the body will describe not a straight course, as when it is impelled by simple impulses, but it will describe curve lines, which differ according as the proportion of the forces differs; excepting however when the forces act in the same direction; or directly opposite to each other, in which two cases, the motion of the body will always be rectilinear.

Imagine that the body *A*, fig. 4. Plate III. is impelled from *A* towards *H*, with such a force as by itself would enable it to run over the equal spaces *AB*, *BF*, *FG*, &c. in equal portions of time; for instance, each of those distances in one minute. Imagine likewise that an attractive (consequently, an accelerative) force, continually draws the same body *A* towards the centre *C*, in such a manner

es by itself would enable it to run over the unequal spaces AI, IK, KL, LM, in equal portions of time, viz. a minute each.

Now, the joint action of both those forces, must compel the body A to run the compound and curvilinear course A N O P, &c.—Through B draw the line B C, that is, in the direction of the centre of attraction;—through I draw I N parallel to A B; and it is evident, from what has been said above, that at the end of the first minute the body will be found at N. Now if at this period the attractive force ceased to act, the body would run on in the direction N R, by the first law of motion. But since the attractive force continues to act, the body at the end of the second minute will be found at O; for the like reason, at the end of the third minute it will be found at P, and so on. The course then ANOP is not straight; but it consists of the lines AN, NO, OP, &c. forming certain angles with each other,

If instead of finding the place of the body at the end of every minute, we had determined its place at the end of every half minute; then each of those lines AN, NO, &c. would have been resolved into two lines containing an angle. And in the same manner, if we had determined the situation of the body at the end of every thousandth part of a minute, each of the lines AN, NO, &c. would have been resolved into a thousand lines inclined to each

other; but since the attractive force acts not at intervals, but constantly and unrenittedly; therefore, the real path of the body is a polygonal course, consisting of an infinite number of sides; or more justly speaking, it is a continue curve line, which passes through the points A, N, O, P, &c. as is shewn by the dotted line.

The curvature of the path ANOP of a body, which is acted upon at the same time by an equable, and by an accelerative force, varies according to the proportion of the two forces. Thus if the equable impulse be increased or diminished, in such a manner as by itself would enable the body to pass over spaces longer or shorter than AB, BF, FG, &c. in the like equal portions of time, as were supposed above, the attractive force remaining the same; then the curvature of the path will be increased or diminished accordingly, as is shewn in fig. 5 and 6. P. III.

When the two forces are in a certain ratio to each other, then the course or path of the body is a circle; in other proportions within a certain limit, the path becomes elliptical, or an oval more or less extended; and in other proportions beyond that limit the path becomes an open curve, or such as never returns to itself. Such curves are called *parabolas* or *hyperbolas*, and their properties, as well as those of the *ellipsis*, are described by all the writers on conic sections,

In fig. 4, the centre of attraction C, has been placed not very far from the direction AH of the equable force. But when this centre is very far from it, the right lines CB, CF, CG, &c. will become nearly parallel, and in many cases, they may without error, be considered as being actually parallel.

A case of this sort is represented in fig. 7. Plate III. where the centre of attraction is so remote from AG, that the right lines BC, FC, GC, &c. which proceed from it, are not to be distinguished from parallel lines. In this case, if the spaces AI, AK, AL, &c. be as the squares of the times, viz. as the squares of one minute, of two minutes, of three minutes, &c.; whilst the spaces AB, AF, AG, &c. or their equals IN, KO, LP, &c. be simply as the times, then the curve or path of the body, ANOP, is a sort of curve called a *parabola*, which is more or less open according as the projectile or equable force is more or less powerful. And such is the path which is described by all bodies that are projected obliquely near the surface of the earth, viz. cannon balls, stones thrown by the hand or other engine, and in short by all sorts of projectiles; excepting however that deviation from the parabolic curvature, which is occasioned by the resistance of the air; and which in certain cases is very considerable. For near the surface of the earth, the spaces described by descending bodies, are as the squares
of

of the times, (according to what has been shewn in Chap. V.) and the centre of attraction is about 4000 miles below the surface. (1.)

The

(1.) The admirable doctrine of curvilinear motion deserves the greatest attention of the philosopher, since it unfolds the grandest phenomena of nature. It comprehends almost all the movements which take place in the world. It measures and ascertains every particular relative to the motions of cœlestial bodies.—It leads the human mind, through safe paths, to the investigation and knowledge of the most complicated appearances, and the most abstruse subjects. I shall, therefore, in this place endeavour to explain this doctrine in as concise and comprehensive a manner, as the nature of the subject, and the limits of the work, may seem to allow,

Of Equable Motion in Circular Orbits.

A *centripetal force*, in its full meaning, is that whereby a body in motion is continually drawn from its rectilinear course, towards some centre. This force may likewise be the action of a string holding the body; or it may be its coherence with another revolving body, or it may be the gravitating power, &c.

A *centrifugal force* is the re-action or resistance, which a moving body exerts to prevent its being turned out of its way, and whereby it endeavours to continue its motion in the same direction; and as re-action is always equal and contrary to action, so is the centrifugal to the centripetal force. The centrifugal force arises from the inertia of matter; for the body that moves round a centre, would fly off in the direction of the last moment, or last particle
of

The paths of bodies that move round a centre of attraction, are possessed of several remarkable properties ;

of its curvilinear course, viz. in a tangent to the curve, were the action of the centripetal force to be suspended. The equality of the two forces, viz. of the centripetal in opposition to the centrifugal force, may be more easily conceived, by imagining that a revolving body is detained within its circular orbit by a string ; for this string must equally endeavour to draw the body towards the centre of attraction, and the centre of attraction towards the body.

Since the centripetal force is proportionate to the space which the body describes in a given time by the action of that force, it is evident that the centripetal as well as the centrifugal force, may be represented by the nascent lines BC, *bc*, fig. 8. Plate III. for whilst the body describes the infinitely small tangent AB, the space which the centripetal force compels it to pass through in the same time, is equal to BC.

N.B. The lines BC, *bc*, as well as AB, *A b*, are drawn so large, merely for the sake of illustration ; whereas by nascent or evanescent lines, we mean lines of the same nature, but indefinitely small, and near the point A.—The same thing must be understood of other lines or quantities, which are nascent or evanescent in the following propositions.

Proposition I. *In a very small arch, the sine, the tangent, the chord, and the arch itself, are to each other nearly in the ratio of equality.*

The right-angled triangles ABE, and ACD, fig. 9, Plate III. are similar ; therefore, $AE : AD :: BE : CD$, when the arch BFD, or angle BAD becomes very small,

or

perties; that is their periods, their velocities, their distances from the centre of attraction, &c. follow certain invariable laws, the knowledge of which is exceedingly useful in the investigation of natural phenomena.

or smaller than any given quantity, the point E will approach the point D indefinitely near; so that the difference between AE and AD will almost vanish, and of course the difference between BE (the sine) and CD (the tangent) will likewise nearly vanish; viz. the sine and the tangent will become nearly equal. And since the chord BD, and the arch BFD, are each of them longer than the sine, and shorter than the tangent; therefore in very small arches, the sine, the tangent, the chord, and the arch itself, are nearly equal.

Prop. II. *In a circle the evanescent, or infinitely small subtenses of the angle of contact, are as the squares of the conterminal arches.*

In fig. 8. Plate III. BC, and bc drawn perpendicular to the tangent Ab , are the subtenses of the angle of contact bAc , made by the tangent Ab , and the circumference ACD; which subtenses must be imagined to be very near the point A; in which case we shall prove them to be to each other as the squares of the conterminal arches AC, A c .

In consequence of the parallelism of the lines AD, BC, bc , and of Ab , mC , nc ; the line BC is equal to Am , and bc is equal to An . (By Eucl. p. 8. B. VI.) $AD : AC :: AC : Am$; and $AD : A c :: A c : $An$$; therefore $AD \times Am = \overline{AC}^2$; and $AD \times An = \overline{A c }^2$. Hence we have $\overline{AC}^2 : \overline{A c }^2 :: Am \times AD : An \times AD :: Am : An :: BC : bc$.

Here

phenomena. Those laws will be found mathematically deduced from a few well established principles,

Here AC, A c, may be taken for the arches as well as for the chords which subtend those arches; since, by the preceding proposition, they are nearly equal.

Corollary. Since $AD : AC :: AC : Am (= BC)$; we have $BC = \frac{AC^2}{AD}$.

Prop. III. *In the first or nascent state of circular motion, the projectile force infinitely exceeds the centripetal force.*

In fig. 10. Plate III. the circle ACD represents the orbit of the body A, moving equably along the said circumference; viz. the body A is impelled by a projectile force, in the direction AH perpendicular to AN, and is at the same time constantly acted upon by an attractive force in the direction towards the centre N; those two forces being so adjusted, or being in such proportion to each other, as to keep the body in the circular orbit ACDA.

In the very small arch AC, the line AB is to the line Am (= BC) as the force of projection is to the attractive, or centripetal, force, at the distance AN; for whilst AB represents the equable movement which arises from the projectile force in a certain time, BC represents the deviation from that course, or the force whereby the body is drawn towards the centre of attraction in the same time.

Now, by the preceding proposition $BC (= Am) : AC :: AC : AD$; and when the arch AC becomes extremely small, or is in its nascent state, then the diameter AD becomes infinitely greater than AC; and of course AC, or AB (which by p. 1. is nearly equal to it) becomes infinitely greater

ples, in the note immediately under this paragraph; But the principal of them will be proved experimentally in the sequel. For the present, the reader

greater than BC, or Am ; viz. the projectile force infinitely greater than the central force.

In order to compare, and to demonstrate with more expedition, the proportions relative to the velocities, the forces, &c. of bodies revolving equably in different circular orbits, as ACD . and ILO , fig. 10, it will be useful to substitute letters instead of those particulars, and whilst the capital letters are applied to the body A moving in the circular orbit ACD , the small letters of the same name will denote the same things with respect to the body I moving in the circular orbit ILO . Therefore let

F, f , stand for the central forces.

V, v , for the circular velocities, or for any arches AC , IL ; since in equable motions the spaces passed over in a given time are as the velocities.

T, t , stand for the periodical times.

D, d , for the diameters, and

P, p , for the peripheries or circular orbits.

The meanings of those letters will be easily remembered, since they are the initials of what they are meant to represent.

Prop. IV. *The central forces are as the squares of the velocities directly, and as the diameters inversely.*

By the substitution of the above-mentioned letters, the equation of the corollary to prop. 2d. becomes $F = \frac{V^2}{D}$, or

$$f = \frac{v^2}{d}; \text{ hence } F : f :: \frac{V^2}{D} : \frac{v^2}{d}.$$

Prop.

reader would do well if he fixed in his mind two of those laws, which are as follows, and whose use is very extensive.

1st.

Prop. V. *In different circular orbits the central forces are as the diameters directly, and as the squares of the periodical times inversely.*

In equable motions the velocity is expressed by the quotient of the space divided by the time; and in circular motion the periphery is the space; hence $V = \frac{P}{T}$. Since, by p. iv. $F = \frac{v^2}{D}$; we have $FD = V^2$, and $\overline{FD}^{\frac{1}{2}} = V = \frac{P}{T}$; therefore $\overline{FD}^{\frac{1}{2}} \times T = P$; and $FDT^2 = P^2$. But the periphery of any circle is equal to $3,1416$ multiplied by the diameter; therefore $P^2 = \overline{3,1416}^2 \times D^2 = FDT^2$; and $\overline{3,1416}^2 \times D = FT^2$; hence we have the force $F = \frac{\overline{3,1416}^2 \times D}{T^2}$ for the circular orbit ACD fig. 10, and $f = \frac{\overline{3,1416}^2 \times d}{t^2}$ for any other circular orbit as ILO; therefore $F : f :: \frac{\overline{3,1416}^2 D}{T^2} : \frac{\overline{3,1416}^2 d}{t^2} :: \frac{D}{T^2} : \frac{d}{t^2}$.

Prop. VI. *When the revolving bodies describe equal areas in equal times, then the central forces are as the cubes of the diameters.*

In this case the area is represented by VD , which being equal to the other area vd , we have $V : v :: d : D$, and $V^2 : v^2 :: d^2 : D^2$. But (by prop. IV.) $F : f :: \frac{V^2}{D} : \frac{v^2}{d}$; hence $\frac{V^2}{D} : \frac{v^2}{d} :: V^2 d : v^2 D$; therefore $F : f :: d^2 d : D^2 D :: d^3 : D^3$.

Prop.

1st. *When bodies revolve in equal circles, the central forces are as the squares of the velocities; or a double projectile force balances a quadruple force of centripetal*

Prop. VII. *When the periodical times are equal, the central forces are as the radii, viz. as the distances from the centre of attraction, and vice versa.*

When $T = t$, the analogy of Prop. V. (viz. $F : f :: \frac{D}{T^2} : \frac{d}{t^2}$) becomes $F : f :: D : d :: \frac{D}{2} : \frac{d}{2}$ (viz. as the radii); the converse of this proposition is also evident, viz. that when $F : f :: D : d$, the periodical times must be equal.

Prop. VIII. *When the diameters are equal, the central forces are as the squares of the velocities.*

For (by Prop. IV.) $F : f :: \frac{V^2}{D} : \frac{v^2}{d}$; therefore when $D = d$, this analogy becomes $F : f :: V^2 : v^2$; viz. when the circles are equal, or in the same circle, the forces are as the squares of the velocities.

Prop. IX. *When the diameters, or distances, and of course the circles, are equal, the central forces are inversely as the squares of the periodical times.*

Since in that case the analogy of Prop. V. (viz. $F : f :: \frac{D}{T^2} : \frac{d}{t^2}$) becomes $F : f :: \frac{1}{T^2} : \frac{1}{t^2} :: t^2 : T^2$.

Prop. X. *When the diameters are equal, the periodical times are inversely as the velocities.*

It appears from Prop. V. that $V : v :: \frac{P}{T} : \frac{p}{t} ::$

$\frac{D}{T} : \frac{d}{t}$ (since the diameters of circles bear to each other the

same

centripetal attraction. For instance, if a body, which is impelled with a certain velocity, and is attracted with

same proportion as their peripheries). Now when $D = d$, then the preceding analogy becomes $V : v :: \frac{1}{T} : \frac{1}{t}$; or $T : t :: v : V$.

Prop. XI. *When the velocities are equal, the forces are inversely, as the diameters.*

For in that case, the analogy of p. IV. (viz. $F : f :: \frac{V^2}{D} : \frac{v^2}{d}$) becomes $F : f :: \frac{1}{D} : \frac{1}{d}$; or $F : f :: d : D$.

Prop. XII. *When the velocities are equal, the periodical times are as the diameters; or as the peripheries, which is the same thing.*

By prop. V. $V : v :: \frac{P}{T} : \frac{p}{t}$; and when $V = v$, then $\frac{P}{T} = \frac{p}{t}$, or $P t = p T$; whence we have $T : t :: P : p :: D : d$.

Prop. XIII. *When the central forces are equal, the periodical times are as the square roots of the distances, or of the diameters.*

In prop. V. it has been shewn that $F : f :: \frac{D}{T^2} : \frac{d}{t^2}$. Now when $F = f$; then $\frac{D}{T^2} = \frac{d}{t^2}$; or $D t^2 = d T^2$ which gives the analogy $T^2 : t^2 :: D : d$; and of course $T : t :: D^{\frac{1}{2}} : d^{\frac{1}{2}}$.

Prop. XIV. *When the central forces are equal, the squares of the velocities are as the distances; and the periodical times are as the velocities.*

By prop. IV. $F : f :: \frac{V^2}{D} : \frac{v^2}{d}$; and when $F = f$;

with a certain central force, describe a circle round the centre of attraction; then if the velocity be doubled, or tripled, the attractive force must be
four

then $\frac{V^2}{D} = \frac{v^2}{d}$ and $V^2 d = v^2 D$; which gives the analogy $V^2 : v^2 :: D : d :: \frac{D}{2} : \frac{d}{2}$ (the distances being the halves of the diameters). Also by p. XIII. $T : t :: D^{\frac{1}{2}} : d^{\frac{1}{2}}$, and by the last analogy, $V : v :: D^{\frac{1}{2}} : d^{\frac{1}{2}}$; therefore $T : t :: V : v$.

Prop. XV. *When the central forces are inverfely as the fquares of the diameters, or of the diftances; then the fquares of the periodical times are as the cubes of the diftances.*

Imagine the central forces to be as fome power, m , of the diftances; viz. $F : f :: D^m : d^m$. Now by prop. V. $F : f :: \frac{D}{T^2} : \frac{d}{t^2}$; therefore $D^m : d^m :: \frac{D}{T^2} : \frac{d}{t^2} :: D t^2 : d T^2$; and $\frac{D^m}{D} : \frac{d^m}{d} :: t^2 : T^2$; or $D^{m-1} : d^{m-1} :: t^2 : T^2$; and $D^{\frac{m-1}{2}} : d^{\frac{m-1}{2}} :: t : T$; hence $D^{\frac{1-m}{2}} : d^{\frac{1-m}{2}} :: T : t$.

Now when the forces are equal then the power, m , vanifhes, or $m = 0$, and then the laft analogy becomes $D^{\frac{1}{2}} : d^{\frac{1}{2}} :: T : t$, which is the fame thing as was fhewn in prop. XIII. When the forces are as the diftances, then m is the firft power, or $m = 1$, and in that cafe the above-mentioned analogy becomes $D^0 : d^0 :: 1 : 1 :: T : t$, and of courfe $T = t$, which is the cafe of prop. VII.—Laftly, when the forces are inverfely as the fquares of the diftances, then $m = -2$ and the above-mentioned analogy, becomes $D^{\frac{3}{2}} : d^{\frac{3}{2}} :: T : t$; or $D^3 : d^3 :: T^2 : t^2$; or $\frac{D^3}{2} : \frac{d^3}{2} :: T^2 : t^2$.

The

four or nine times as strong as it was before, in order to let the body move in the same circle.

2d. *When bodies move in unequal circular orbits,*
/o

The planets of our solar system follow this grand law of nature. The squares of their periodical times are as the cubes of their distances from the common centre of attraction, which is very near the centre of the sun, as will be shewn hereafter; and thus Newton's hypothesis of mutual and universal attraction amongst the bodies of the universe is shewn to be so consonant with the strictest mathematical reasoning, and with all the appearances, that none but the ignorant can refuse their assent to it.

This doctrine of circular movements, which I have exhibited in 15 propositions, might have been condensed into a narrower compass, had not my principal object been to render the comprehension of it easy to the reader; I have been taught by experience, that in many instances it is far more laborious to deduce every particular case from one comprehensive proposition, than to read a particular proposition for every single case.

Having, in the preceding propositions, stated the proportions between the forces, the velocities, and the periodical times, of bodies that revolve in circular orbits; it is now necessary to render those propositions practically useful, by shewing in what manner they may be employed for the determination of any particular case; since it has already been remarked, that the knowledge of the proportion which certain things bear to each other, will not enable us to determine any absolute quantity, unless some of the particulars be previously determined by means of actual experiments.

so that the squares of the times of their revolutions are as the cubes of their distances from the centres of those circles, then the central forces are inversely as the squares of the distances ; and vice versa.

The

Therefore in order to render the expressions of the measures of the above-mentioned forces, velocities, &c. more easy and concise, we shall endeavour to involve in them only one unknown quantity, viz. a certain power of the radius or distance of the revolving body from the centre of attraction ; for when this quantity becomes known either from experiments or by deduction from other known quantities, we may thereby easily determine all the other particulars.

I. The attractive force is measured by the velocity which may be uniformly generated in a given time, which time we shall call one, (meaning one second, or one minute, or, in short, the unity of any other division of time that may be used in the computation,) and shall express this force or velocity by r^n , (viz. an indeterminate power n of the radius r of the orbit).

II. It has been shewn in page 67, that a body, which begins to move from rest and proceeds towards a centre of attraction, will at the end of any given time acquire such velocity as would enable it to move equably through twice that space in an equal portion of time, if the action of the attractive force were suspended. Therefore the distance through which a body will descend towards the centre of attraction in the above-mentioned time one, is $\frac{1}{2} r^n$.

III. Hence if A C fig. 10; Plate III. be an arch described in a certain time 1, the distance A m, which the body would

The foregoing theory of curvilinear motion is very extensive, since it applies to a great variety of terrestrial as well as celestial phenomena. But in the

would descend through towards the centre of attraction N in the same time, will be $= \frac{\frac{1}{2} r^n T^2}{1^2}$; for since the spaces which are thus descended, as the squares of the times, we have $1^2 : T^2 :: \frac{1}{2} r^n : \frac{\frac{1}{2} r^n T^2}{1^2}$.

IV. The part AC of the circumference, which is described by the body in the time T, is $= \frac{T r^{\frac{n+1}{2}}}{1}$; since in the circle ACD, we have $\overline{AC}^2 = AD \times Am$ (Eucl. p. 8. B. VI.) $= 2AN \times Am = AN \times 2Am = \frac{T^2 r^n}{1^2} = \frac{T^2 r^{n+1}}{1^2}$; therefore $AC = \frac{\sqrt{T^2 r^{n+1}}}{1} = \frac{T r^{\frac{n+1}{2}}}{1}$.

V. The velocity with which the body moves in the circle is $= r^{\frac{n+1}{2}}$. By prop. IV. the square of the velocity is as the product of the diameter, or of the radius, multiplied by the force; and according to the above-mentioned notation, (§. I.) the force is expressed by r^n ; therefore the square of the velocity is $= r r^n = r^{n+1}$, and of course the velocity itself is expressed by $r^{\frac{n+1}{2}}$.

VI. The periodical time, or time of a whole revolution, is $= 2\pi r^{\frac{1-n}{2}}$ (π being $= 3,1416$, &c. that is the circumference of a circle whose diameter is one.) For since

the calculation of the particulars which relate to those phenomena, certain circumstances generally interfere,

(by §. IV.) $\frac{T r^{\frac{n+1}{2}}}{1}$ is the part of the orbit which is described in the time T , the part which is described in the time 1 , must evidently be $r^{\frac{n+1}{2}}$. Then, the spaces described with a uniform motion being as the times, it will be $r^{\frac{n+1}{2}} : 2rc (= \text{the whole circumference}) :: 1 : 2cr^{\frac{1-n}{2}}$.

VII. The space through which the body must descend towards the centre of attraction, in order to acquire a velocity equal to that with which it revolves, is equal to half the radius, viz. $\frac{1}{2}r$. For in Chap. V. it has been shewn, that the spaces described by descending bodies are, as the squares of the times, or of the velocities. It has also been shewn (§. II.) that the velocity r^n is acquired by a descent through $\frac{1}{2}r^n$. At present we wish to know how low a body must descend, to acquire a velocity equal to $r^{\frac{n+1}{2}}$ (§. V.) hence we say, as the square of r^n is to the square of $r^{\frac{n+1}{2}}$; so is $\frac{1}{2}r^n$ to a fourth proportional;

$$\text{viz. } r^{2n} : r^{n+1} :: \frac{1}{2}r^n : \frac{\frac{1}{2}r^{2n+1}}{r^{2n}} = \frac{\frac{1}{2}r^{2n}}{r^{2n}} = \frac{1}{2}r.$$

Thus we have expressed the measures of the velocities, periodical times, &c. in a general yet simple manner. They may be applied to any attractive power, and to any periodical revolution; the only quantity which needs be known,

interfere, which render the result of the calculations somewhat different from the observations; that is, of the experiments. In terrestrial affairs, the

known, being the value of r'' . But for the sake of illustration, we shall now apply it to the force of terrestrial gravity; in which case it is known, that a body near the surface of the earth, will descend from rest 16,087 feet in the first second of time, (which is the time 1); therefore, $\frac{1}{2} r'' = 16,087$ feet, and $r'' = 32,174$. (§. II.) Hence by substituting those values for $\frac{1}{2} r''$ and r'' respectively, the above-mentioned measures will be expressed in known terms.

Example 1. The velocity of a body that revolves round the earth but near the surface of it, is (by §. V.) $r'' \frac{r + 1}{2}$; which, by substituting, 32,174 for r'' , becomes $\sqrt{32,174 r}^{\frac{1}{2}}$; and this becomes (since the semi-diameter or radius r of the earth is known to be nearly 21000000 feet) $\sqrt{32,174 \times 21000000}^{\frac{1}{2}} = 25993,3$ feet per second; so that a body moving with that velocity, would revolve continually round the earth; that velocity being just sufficient to balance the force of gravity; but this velocity is about 30 times as great as the initial velocity of a cannon ball.

N. B. No notice of the resistance of the air has been taken in this example, or will be taken in the following examples of this note.

The periodical time of the same body under the same circumstances, is (by §. VI.) $2 \pi r \frac{1 - n}{2} = 2 \pi \frac{r^{\frac{1}{2}}}{r''^{\frac{1}{2}}} = 2 \times$

$$3,1416 \times \frac{4582,5}{5,67} = 5087'',5; \text{ or } 1 \text{ hour, } 24', 47'',5.$$

the resistance of the air is one of the principal obtruders. The movements of the cœlestial bodies are

Example 2. By prop. IX. when the distances are equal or in the same circle, the central forces are inversely as the squares of the periodical times; and, by the preceding example, the velocity which near the surface of the earth is equivalent to gravity, is $= 25993,3$ feet per second. Therefore, we say as the square of the earth's diurnal rotation round its axis, is to the square of the periodical time of the body mentioned in the preceding example, (viz. of $1^{\text{h}} 24' 47''$, 5, or nearly $85'$); so is the force of gravity (which we shall call 1) to the centrifugal force of bodies near the equator of the earth; viz. $2073600' (= \text{the square of 24 hours}) : 7225' :: 1 : 0,003485 = \text{the centrifugal force near the equator}$; viz. the force by which bodies that are near the equator, are attracted towards the centre, is to the force with which they endeavour to fly off, in consequence of the earth's diurnal rotation round its axis, as 1 is to 0,003485; or as 1000000 to 3485; viz. the former is almost 300 times more powerful than the latter.

By this means we may determine the centrifugal force of bodies in different latitudes; for as the earth turns round its axis, it is evident that those bodies on the surface of it, which lie nearer to the axis, or, which is the same thing, are nearer to the poles, perform circles smaller than those which lie nearer to the equator; though they are all performed in the same time, viz. 24 hours. Hence (by prop. VII.) the periodical times being equal, or the same, the central forces are as the radii of the circles, and as in different latitudes the radii are equal to the cosines of the latitudes, therefore, as the radius is to the cosine of a given latitude,

are generally influenced by more than one centre of attraction. Thus the moon is attracted by the earth and likewise by the sun. The planets are attracted

latitude, so is the centrifugal force of bodies situated at the equator, to the centrifugal force of bodies at that given latitude. Now as the cosines grow shorter and shorter, the nearer they come to the poles, so the tendency of bodies to fly off from the surface of the earth is greatest at the equator, but it diminishes as you approach the poles; and hence we see why the earth has been found by means of undoubted measurements and other observations, to be an oblate spheroid, whose polar diameter is the shortest. And this furnishes a strong evidence of the earth's daily rotation about its axis.

Example 3. The mean distance of the moon from the centre of the earth is, 1267200000 feet, or about 60 semi-diameters of the earth. Also the force of gravity at different distances, is inversely as the squares of the distances, and the radius of the earth is 21000000 feet; therefore, as the square of 1267200000, is to the square of 21000000, so is the force of gravity at the surface of the earth, to the force of gravity at the distance of the moon, viz. 160579 5840000000000 : 441000000000000 :: 1 : 0,000274; so that the force of gravity at the surface of the earth, is to the force of gravity at the moon as 1 is to 0,000274; or as 1000000 to 274. And since near the earth falling bodies pass over 16,087 feet in the first second of time; therefore, we say, 1000000 : 274 :: 16,087 : 0,0044 of a foot; which shews that the moon, should its velocity cease at once, would fall towards the earth, and in the first second of time would descend through not more than $\frac{44}{1000000}$ ths of a foot.

Farther.

attracted by the sun, and likewise by each other, &c.

On this account we might now extend our examination to the cases in which two or three, or more

Farther. By prop. XV. when the central forces are inversely as the squares of the diameters, then the cubes of the distances are as the squares of the periodical times. Therefore the distance of the body, which circulates near the surface of the earth (Example 1.) being one semidiameter of the earth, and the distance of the moon being 60 semidiameters; also the period of the former being 84',8 we may find the period of the latter by saying $1^3 : 60^3 :: 84',8^2 :$ to the square of the moon's period; viz. $1 : 216000 :: 7191 : 1553256000$; the square root of which, viz. 39411'',3 or 27 days 8 hours 51',3, is the period of the moon's revolution round her orbit, which is nearly equal to what the astronomers reckon it, viz. $27^d 7^h 34'$; and it would have come out exactly like it, had the distances been stated with exactness; and had we likewise taken into the account certain circumstances, which interfere with that period, which however we have purposely avoided in this example, for the sake of brevity.

Similar calculations may be instituted with respect to all the planets of our solar system, and the result of the calculations will be found to coincide wonderfully well with the appearances; which, as we have already remarked, is a strong confirmation of the Newtonian theory of universal gravitation.

Example 4. Let a ball of one pound weight be fastened to a string 2 feet long, and be whirled about a centre so as to describe each revolution in half a second. In this case the

more centres of attraction act upon the same body ; but this investigation we shall omit on two accounts, viz. first, because the subject is too intricate and

the orbit or circumference of the circle is $4 \times 3,1416 = 12,5664$. The velocity of the ball is 25,1328 feet per second. In order to determine the centrifugal force of the ball thus revolving, viz. the force with which the string is stretched by it, compared with the force of gravity (which is = 1), we make use of the analogy of prop. V. (viz. $\frac{D}{T^2}$;

$\frac{d}{t^2} :: F : f$) which, by substituting $2r$ for D ; $2cr \sqrt{\frac{1-n^2}{2}}$ for T^2 ; 4 for d ; 0'',25 for t^2 ; and 1 for F ; becomes $\frac{r^n}{2c^2} \left(= 2r \div 2cr \sqrt{\frac{1-n^2}{2}} \right) : \frac{d}{t^2} :: 1 : f = \frac{2dc^2}{rnt^2} =$

$\frac{dc^2}{\frac{1}{2}rnt^2}$, equal to $\frac{4 \times 9,86}{16,087 \times 0,25} = \frac{39,44}{4,02175} = 9,8$ which is

the measure of the central force of the body in question; this force therefore is to the force of gravity as 9,8 to 1; so that since the body weighs one pound when quiescent, viz. it stretches the string with the weight of one pound; therefore when revolving according to the supposition, it will stretch the string with the force of 9,8 pounds.

Now this central force may be called centripetal or centrifugal, according as it is applied to the tenacity of the parts of the string, or to the force of the body; so that the body is said to be retained by a centripetal force 9,8 times as great as the force of terrestrial gravity; or it may be said that the centrifugal force of the revolving body stretches the string as much as if a weight of 9,8 pounds were simply suspended to it.

and extensive ; and secondly, because in most natural phenomena, the disturbing cause which arises from the action of a second or a third, or in general

Of the motion of bodies about a centre of attraction, but in curves differing from circles.

It has been sufficiently shewn that a certain determinate velocity is required to confine the movement of a body in a circular orbit round a centre of attraction ; whence it follows, that with a greater or a lesser velocity bodies will move in curve lines different from circles. Those curves appear to be the conic sections ; and since, strictly speaking, the circle is likewise a conic section, therefore it may be concluded, that in general the movements of bodies round any centre of attraction are performed in curves of the conic kind, provided the bodies do not meet with any obstructing medium, or other attraction, in their way ; for under such circumstances, their paths may degenerate into spirals, or other curves of a more intricate nature.

The movements of the celestial bodies are not strictly circular, though they do not deviate much from that figure ; excepting however the comets which move either in very eccentric ellipses, or else in parabolas or hyperbolas ; and therefore in the last two cases they can never return to the same parts of the heavens ; but they must continually recede from the common centre of attraction, which, in our solar system, is not far from the centre of the sun.

With respect to the theory of circular movements, I have endeavoured to demonstrate the principles, and to illustrate the practical operations in a manner sufficiently extensive ; being persuaded that if that branch of compound motion be well understood, the reader (provided he is acquainted with the

neral of more than one centre of attraction, is not very considerable; yet in the course of this work, the method of taking the above-mentioned circumstances

the principal properties of the conic sections) will easily comprehend what follows; I shall therefore endeavour to explain the nature of the movements in curves of the conic kind, in a manner more comprehensive and concise.

In Fig. 11. Plate III. ACD represents a circular orbit, Az S represents an elliptical orbit, A r E a parabolic, and AKF an hyperbolic orbit, of bodies moving with certain velocities under the influence of the centre of attraction N, which is the centre of the circle, and the focus of the conic sections.

Let AB, perpendicular to AD, represent the velocity which is necessary to retain the body in the circular orbit, and let this velocity be called 1; for we shall compare the other degrees of velocity with this unity. Also let a body be projected from A in the direction AI with any other degree of velocity n . It is now necessary to determine the nature of the curve which will be described with this other velocity n , or rather it is required to ascertain what the value of n must be in order to produce each particular conic section.

Draw mK parallel to AI, intersecting the circle as well as the other curves. Let AN be denoted by d ; the semi-transverse axis of any of the conic sections, by a ; the semi-conjugate, by b ; and $Am (=BC=Gz=Hr=IK)$ by x . Then the ordinate mC in the circle will be $=\sqrt{2dx - xx}$, but the ordinate mz of the ellipsis, and mK of the hyperbola may be both represented by $\frac{b}{a} \sqrt{2ax \mp xx}^{\frac{1}{2}}$.

The

cumstances into the account, will in many cases be sufficiently pointed out.

It is however proper to observe, that the various circumstances which obstruct or influence the movements

The fluxions of those ordinates are $\frac{d\dot{x} - x\dot{x}}{2d\dot{x} - x\dot{x}}^{\frac{1}{2}}$ and $\frac{b}{a} \times \frac{\dot{a}\dot{x} \mp x\dot{x}}{2a\dot{x} \mp x\dot{x}}^{\frac{1}{2}}$ which fluxions are to each other as the velocities in every point of their respective curves in the direction AI; and in the like proportion are the quantities $\frac{d - x}{2d - x}^{\frac{1}{2}}$ and $\frac{b}{a} \times \frac{a \mp x}{2a \mp x}^{\frac{1}{2}}$ those quantities being the above mentioned fluxions divided by the same quantity, $\frac{\dot{x}}{x}^{\frac{1}{2}}$.

Now when the point in the curve approaches the point A so near as to coincide with it, then Am vanishes, or $x=0$; and the above expressions become $\frac{d}{2d}^{\frac{1}{2}}$ and $\frac{b}{a} \times$

$\frac{a}{2a}^{\frac{1}{2}}$; so that at the point A the velocity which retains the

body in the circular orbit, is to the velocity which retains the body in the ellipsis or the hyperbola, as $\frac{d}{2a}^{\frac{1}{2}} : \frac{b}{a} \times \frac{a}{2a}^{\frac{1}{2}}$

$\therefore d^{\frac{1}{2}} : \frac{b}{a^{\frac{1}{2}}} :: 1 : n$; therefore $n d^{\frac{1}{2}} = \frac{b}{a^{\frac{1}{2}}}$; and $md =$

$\frac{lb}{a}$, or $annl = bb$. When $x = d = AN$, then $2y$ is the

parameter, and (since the parameter is a third proportional to the transverse and conjugate diameters) $2a : 2b :: 2b :$

movements of bodies, are far from being all known, or fully understood. Besides, even those that are known,

$2y$, or $a : b :: b : y = \frac{bb}{a} = \frac{b}{a} \times \sqrt{2ax \mp xx}^{\frac{1}{2}} = \frac{b}{a} \times \sqrt{2ad \mp dd}^{\frac{1}{2}} = \frac{2ab^2d \mp b^2d^2}{aa}^{\frac{1}{2}}$; which equation being squared, becomes $\frac{2ab^2d \mp b^2d^2}{aa} = \frac{b^4}{a^2}$ for the ellipsis, and $\frac{2ab^2d + b^2d^2}{a^2} = \frac{b^4}{a^2}$ for the hyperbola. And being divided by $\frac{b^2}{a^2}$, those expressions become $2ad - d^2 = b^2 = ann$ for the ellipsis, and $2ad + d^2 = b^2 = ann$ for the hyperbola. Therefore

In the ellipsis $\left\{ \begin{array}{l} \text{the femi transverse axis is } a = \frac{d}{2-n^2} \\ \text{the femiconjugate axis is } b = \frac{nd}{2-n}^{\frac{1}{2}} \end{array} \right.$

In the hyperbola $\left\{ \begin{array}{l} \text{the femitransverse is } a = \frac{d}{n^2-2} \\ \text{the femiconjugate is } b = \frac{nd}{nn-2}^{\frac{1}{2}} \end{array} \right.$

Having determined those values of the transverse and conjugate diameters, wherein n is the only indeterminate value, we may, by making certain substitutions instead of n , ascertain what the value of n must be in order to produce one curve or another.

Thus by making $n=1$, each of the above values becomes equal to d ; therefore the two diameters become equal to each other, the curve is of course a circle. And in

known, are mostly fluctuating in the intensity of their actions. Much light has undoubtedly been thrown

in fact the velocity which retains the revolving body in a circular orbit, has been called 1, or unity.

If we make $n = \sqrt{2}^{\frac{1}{2}}$, then $a = \frac{d}{2-nn} = \frac{d}{2-2} = \frac{d}{0}$,

which is an algebraical expression of infinity. And all the other expressions will likewise become infinite; hence, the transverse and conjugate diameters in that case becoming infinite, the curve is the parabola.

If we make n equal to a quantity less than the square root of 2 (viz. less than the square root of twice that velocity which is required to retain the body in a circular orbit;)

then the values $\frac{d}{2-n^2}$ and $\frac{nd}{2-nn^{\frac{1}{2}}}$ viz. of a and b , will be

positive; whereas, by the same substitution, the value $\frac{nd}{nn-2^{\frac{1}{2}}}$ becomes impossible; which shews, that when n

is less than the square root of 2, the curve can only be the ellipsis.

Lastly, if we make n equal to any thing greater than the square root of 2; then the values of a and b for the hyperbola become positive; whereas those for the ellipsis become impossible; hence in this case the curve must be the hyperbola.

We shall conclude this subject with the following general proposition, which, together with its corollaries, is applicable to a variety of natural phenomena.

In all determinate orbits, described by bodies revolving with certain velocities in non resisting mediums, about a centre of attraction, the areas, which are described by a straight line connecting

thrown on this subject by the ingenuity of scientific persons during the two last centuries, yet a great deal still remains to be done, and a vast field of speculation offers itself to the industry of future philosophers.

In

connecting the centre of attraction and the revolving body, lie in one invariable plane, and are always proportional to the times in which they are described.

Imagine the time to be divided into equal particles, and that a body moving round the centre of attraction N, fig. 12, Plate III. runs over the space AB in the first particle of time. It is evident that, were the body left to itself, it would proceed straight to H, describing BH, equal to AB, in the second particle of time; but at B imagine that the body receives a single instantaneous impulse from the centre of attraction N in the direction BN, sufficient to change its direction from BH to BC. Through H draw CH parallel to BN, which will meet BC in C; and, agreeably to the laws of compound motion, at the end of the second particle of time, the body will be found at C in the same plane with the triangle ANB. Draw the lines NC, NH, and the triangle NBH will be equal to the triangle NBC, since they stand on the same base NB, and between the same parallels NB, CH (Eucl. p. 37, B. I.) It will likewise be equal to the triangle ABN, since they have equal bases and the same altitude (Eucl. p. 1. B. VI.) By the very same mode of reasoning it may be proved, that, if the centripetal force act upon the body at the end of each successive particle of time, so as to let the body describe the spaces CD, DE, EF, &c. those spaces will all lie in the same plane; the triangles ANB, BNC, CND, DNE, &c.

In the present state of the world the improvements of science seldom die with individuals. The accumulation of knowledge by leading the understanding, and by furnishing tools to the senses, promotes the discovery of farther truths, and the inexhaustible

will be all equal, and will be described in equal times. Consequently two or three, or any number of them, will be described in two or three, or the like number of particles of time, viz. they are as the times.

Now imagine that those triangles are infinitely increased in number, and diminished in size; then the polygonal path ABCDEF, will become a continuatè curve; for the constant action of the centre of attraction will be continually drawing the body away from the direction of the tangent at every point of the curve. And it is evident that the sectoral areas of the said curve, or number of infinitely small triangles, must be proportional to the times in which they are described, and that the curve must lie in one immoveable plain.

Corollary 1. The velocities in different parts of the orbit are inversely as the perpendiculars dropped from the centre of attraction on the tangents to the orbit at those parts or points. For since the velocities are as the bases AB, BC, CD, &c. of equal triangles, they must be inversely as the heights of those triangles, (Eucl. p. 15, B. VI. and p. 38, B. I.) which are the same as the perpendiculars dropped from the centre N, on the tangents to the orbit at those points.

Corollary 2. The times in which equal parts, or arches of the orbit are described, are directly as those perpendiculars to the tangents. For when the arches, or bases of the triangles, are equal, the triangles are as their altitudes; that is,

exhaustible fund of nature offers on all sides innumerable objects of investigation to the inquisitive mind.

is, as the above-mentioned perpendiculars. But they are likewise as the times; therefore, &c.

Corollary 3. If, by drawing lines parallel to the chords AB, BC, of any two contiguous and evanescent arches described in equal times, the parallelogram be completed, the diagonal BG, when produced, will pass through the centre of attraction N, which proves the converse of the proposition; viz. that *when the areas, which are described by a straight line, connecting a moving body and a certain point, are proportional to the times in which they are described, then the body is under the influence of a centripetal force tending to that point.*

Corollary 4. *In every point of the orbit the centripetal force is as the sagitta, or versed sine, of the indefinitely small arch at that point.*—The centripetal force at B is as BG, because BG is equal to CH, and CH is the deviation from the straight direction AH, which has been occasioned by the centripetal force. And the half of BG, viz. BO, is the sagitta, or versed sine, of the indefinitely small arch ABC.

CHAPTER X.

OF THE DESCENT OF BODIES UPON INCLINED PLANES; AND THE DOCTRINE OF PENDULUMS.

Prop. I. **W**HEN a body is placed upon an inclined plane, the force of gravity which urges that body downwards, acts with a power so much less, than if the body descended freely and perpendicularly downwards, as the elevation of the plane is less than its length.

If BD, fig. 1, Plate IV. be an horizontal plane, and a body A be laid upon it, this body will remain motionless; for though the power of gravity, or (which is the same thing) its own weight, draws it towards the centre of the earth, yet the plane DB supports it exactly in that direction; hence no motion can arise.

But if the plane be inclined a little to the horizon, as in fig. 2, Plate IV. then the body will descend gently towards the lower end D. And if the inclination of the plane be increased, as in fig. 3, Plate IV. the body will run down towards D with greater quickness.

In the two last cases; or, in general, whenever the plane is inclined to the horizon, the action of gravity is not entirely but partially counteracted by the plane. For if, from the centre A of the body, in the figures 2 and 3, you draw two lines, viz. AG

per-

perpendicular to the horizon, and AF perpendicular to the plane; the whole force of gravity, which is represented by the line AE, is resolved into two forces; viz. AF and EF, whereof AF being perpendicular to the plane, is that part of the gravitating power which is counteracted by the inclined plane; or that part of the weight of the body which is supported by the plane BD; and EF represents the other part of the gravitating power, which urges the body downwards along the surface of the plane. Therefore the force of gravity which moves the body, is diminished in the proportion of AE to EF. But the triangles AFE, EDG, and BDC, are equiangular, and of course similar (because the angles at F, C, and G are right, and the angle AEF is equal to the angle DEG, by Eucl. p. 15. B. I.; as also equal to the angle DBC, by Eucl. p. 29. B. I.) Hence we have AE to EF, as DB to BC; viz. as the length of the plane is to its elevation, or as the whole force of gravity is to that part of it which urges the body down along the inclined plane*.

Prop. II. *The space which is described by a body descending freely from rest towards the earth, is to the space which it will describe upon the surface of an in-*

* If (by trigonometry) DB be made radius, BC becomes the sine of the angle of inclination BDC; therefore the whole force of gravity is said to be to that part of it which urges a body down an inclined plane, as radius is to the sine of the plane's inclination to the horizon.

clined plane in the same time as the length of the plane is to its elevation, or as radius is to the sine of the plane's inclination to the horizon.

The force of gravity, which urges a body down along the surface of an inclined plane, is diminished by the partial counteraction of the inclined plane ; but its nature is not otherwise changed ; viz. it acts constantly and unremittedly. Hence the velocity of the body is continually accelerated, and the spaces it runs over are also proportional to the squares of the times ; though those spaces will not be so long as if the body descended freely and perpendicularly towards the ground.

Now in order to ascertain how much the space, which is described by a body running down an inclined plane in a certain time, is shorter than the space through which it would descend freely and perpendicularly in the same time, we must recollect what has been proved in page 64, relatively to the spaces, which are described in the same time, by bodies that are acted upon by different central forces ; namely, that in equal times, the spaces are as the forces ; then, since the whole force of gravity is to that force which draws a body down the inclined plane, as radius is to the sine of the plane's inclination. Therefore the space described by a body which descends freely, is to the space which a body will describe on an inclined plane, in the same time as radius is to the sine of the plane's inclination, or as the length of the plane is to its altitude.

Example.

Example. Let the length BD of the inclined plane, fig. 2, Plate IV. be 10 feet, and the elevation BC, 4 feet. It is known from experiment, that in the first second of time, a body will descend freely from rest through 16,087 feet. Therefore, by the rule of three, we say, as 10 feet are to 4 feet, so are 16,087 feet to a fourth proportional, viz. $10 : 4 :: 16,087 : \left(\frac{4 \times 16,087}{10} = \right) 6,435$ feet, which shews

4

that a body running down the inclined plane BD, would pass over little less than six feet and a half, or 6,435 feet, in the first second of time.

Prop. III. *If upon the elevation BC, fig. 4, Plate IV. of the plane BD, as a diameter, the semicircle BEGC be described, the part BE of the inclined plane, which is cut off by the semicircle, is that part of the plane over which a body will descend, in the same time that another body will descend freely and perpendicularly along the diameter of the circle, viz. from B to C, which is the altitude of the plane, or sine of its inclination to the horizon.*

The triangle BEC is equiangular, and of course similar, to the triangle BDC (for the angle at B is common to both, and the angle BEC is by Eucl. p. 31. B. III. a right angle, and therefore equal to the right angle BCD) hence we have BD to BC as BC is to BE. But, by the preceding proposition, the space descended freely and perpendicularly, is to the space run over an inclined plane in the same time, as the length of

the plane is to its elevation, viz. as BD is to BC ; therefore, the space run freely and perpendicularly, is to the space run over the inclined plane, likewise as BC is to BE. And since BC is the space freely descended by a body in a certain time, BE must be the space which is run down by a body on the inclined plane in the same time.

Cor. A very useful and remarkable consequence is derived from this proposition, namely, *that a body will descend from B over any chord whatsoever as BE, or BF, or BG, of the semicircle BEFC, exactly in the same time, viz. in the same time that it would descend freely from B to C.* For if you imagine the inclined plane to be BH instead of BD ; then by this proposition, the body will descend either from B to F, or from B to C in the same time ; and again, if you imagine the inclined plane to be BI, then by this proposition, the body will descend either from B to G, or from B to C, in the same time. And, in short, the same thing may be proved of any other chord of the semicircle.

Prop. IV. *The time of a body's descending along the whole length of an inclined plane, is to the time of its descending freely and perpendicularly along the altitude of the plane, as the length of the plane is to its altitude ; or as the whole force of gravity is to that part of it which acts upon the plane.*

The spaces run over the plane being as the squares of the times, we have the square of the time of passing over BD, fig. 4, Plate IV. to the square of the

the

the time of passing over BE, as BD is to BE. But BD is to BC as BC is BE, viz. BD, BC, and BE are three lines in continue geometrical proportion; therefore (Eucl. p. 20, B. VI.) BD is to BE, as the square of BD is to the square of BC. It has been shewn above, that the square of the time of passing over BD, is to the square of the time of passing over BE, as BD to BE; therefore those squares of the times are to each other as the square of BD to the square of BC; and of course the square roots of these four proportional quantities are likewise proportional (Eucl. p. 22, B. VI.) viz. the time of a body's descending from B to D is to the time of its descending freely and perpendicularly from B to E, or from B to C, as BD is to BC, or as the length of the plane is to its altitude; or (by the 1st proposition of this chapter) as the whole force of gravity is to that part of it which acts upon the plane.

Prop. V. A body by descending from a certain height to the same horizontal line, will acquire the same velocity whether the descent be made perpendicularly, or obliquely, over an inclined plane, or over many successive inclined planes, or lastly over a curve surface.

1st. In page 64, it has been shewn, that the velocity of a body descending freely towards a centre of attraction, is as the product of the attractive force multiplied by the time; and by the preceding proposition it has been proved, that on an inclined plane the force of gravity is diminished in proportion as the time of the body's running down
the

the whole length of the plane, is increased, viz. when the force of gravity is half as strong as it would be in free space, the time is doubled; and when the force is one-third as strong, the time is trebled, &c. therefore the product of the time by the force is always the same; for $\frac{1}{2}$ multiplied by 2 is equal to $\frac{1}{3}$ multiplied by 3, is equal to $\frac{1}{4}$ multiplied by 4, &c. hence the velocity being as that product, must, of course, be always the same, or a constant quantity. For example, suppose, that when the body descends perpendicularly down from B to C, fig. 4, Plate IV. the whole force of gravity acts upon it. Let us call that whole force 1, and let the time employed by the body in coming down from B to C be one minute, then the velocity acquired by that descent is represented by the product of the time by the force, viz. 1 by 1, which makes one. Now when the body descends from the same altitude B, to the same horizontal line DC, over the inclined plane BD, the force of gravity which draws it downwards is diminished; for instance, suppose it to act with a quarter of its original power, then the time of the body's descending from B to D will be four minutes, and the velocity acquired by that descent, being as the product of the force by the time, is as the product of $\frac{1}{4}$ by 4, which is one, or the same as when the body descends perpendicularly down from B to C.

2dly. Suppose that the body descends from the same altitude E to the same horizontal line DC, fig.

fig. 5, Plate IV. along the contiguous inclined planes EF, FG, GD; by the time it arrives at D it will have acquired the same velocity as if it had descended perpendicularly from B to C, or from E perpendicularly down to the horizontal line DC; for, by the first part of this proposition, it will acquire the same velocity whether it descends from E to F or from K to F, and by adding to both the plane FG, it follows that the body will acquire the same velocity whether it descends along the single plane KG, or along the two contiguous planes EF, FG. And by the like reasoning it will be proved, that the body will acquire the same velocity whether it descends along the single plane BD, or along the contiguous two planes KG, GD, or along the contiguous three planes EF, FG, GD, &c.

3dly. If the number of contiguous planes be supposed infinite, and their lengths infinitely small, they will constitute a curve line, like BH; whence it follows, that a body by its descent along the curve line BH, or any other curve, will acquire the same velocity as if it descended perpendicularly from B to C.

Prop. VI. *Let a circle be perpendicular to the horizon, and if two chords be drawn from any two points in the circumference, to the point in which the circle touches the horizon; the velocities which are acquired by the descents of two bodies along those chords, will be as the lengths of the chords respectively.*

It

It has been shewn by the preceding proposition, that a body will acquire the same velocity whether it descends from B to D. fig. 6, Plate IV. or from E to D; D being the point of contact with the horizontal plane GI; and likewise the same velocity will be acquired by descending from C to D, or from F to D; so that the velocities, which are acquired by descending along those chords, are respectively the same as the velocities acquired by descending perpendicularly from E and F to D. And (from what has been shewn in p. 65) those velocities are as the square roots of ED and FD. Now (Eucl. p. 3. B. VI.) AD is to DB as DB is to ED; therefore (Eucl. p. 20. B. IV.) AD is to ED, as the square of AD is to the square of DB, and, for the same reasons, AD is to FD, as the square of AD is to the square of CD. Hence, alternately, AD is to the square of AD, as ED is to the square of BD; and AD is to the square of AD, as FD is to the square of CD; therefore ED is to the square of BD, as FD is to the square of CD; that is, alternately, ED is to FD as the square of BD is to the square of CD; and of course the square root of ED is to the square root of FD as BD is to CD, and as the velocity acquired by descending along BD is to the velocity acquired by descending along CD.

Prop. VII. *If there be two planes of unequal lengths, but equally inclined to the horizon, the times of descent along*

along the whole lengths of those planes will be as the square roots of their lengths respectively.

Let BD and EF, fig. 7, Plate IV. be two planes of unequal lengths, but equally inclined to the horizon; and it follows from prop. IV. of this chapter, that the time of descent along the plane BD is to the time of the perpendicular descent along BC, as BD is to BC; also that the time of descent along EF is to the time of descent along EC, as EF is to EC. The times of the perpendicular descents along BC and EC are as the respective square roots of BC and EC (see page 65.) Now the triangles BDC and EFC being equiangular, and therefore similar (Eucl. p. 4. B. VI.) we have BC to EC as BD to EF, and of course the square root of BC is the square root of EC, as the square root of BD is to the square root of EF; viz. as the time of descent along BD is to the time of descent along EF.

Cor. The same thing must be understood (as it may easily be derived from the above proposition) of two or more contiguous planes similarly situated, as BID, EHF; and likewise of two curve surfaces that are similar and similarly situated; since those curves may be conceived to consist of an infinite number of planes similarly situated.

Thus much will suffice for the present with respect to the properties of inclined planes, in which

we

we have supposed the bodies to be spherical, and the planes as well as the bodies to be perfectly smooth and not obstructed, either by friction or by the resistance of the air. We shall now explain the properties of pendulums or pendulous bodies; a pendulum being a body hanging at the end of a string, like A, fig. 8, Plate IV. and moveable about a fixed *point of suspension* C. A pendulum however may consist of a single body suspended without any string, such as a rod of wood or other matter suspended by one end, &c. but in the following propositions a pendulum must be understood to be according to the former definition, viz. a body suspended at the end of a string, &c. and the string must be supposed to be void of weight, as also to move with perfect freedom about the point of suspension, unless the contrary be mentioned.

Prop. VIII. *If a pendulum be moved to any distance from its natural and perpendicular direction, and there be let go, it will descend towards the perpendicular, then it will ascend on the opposite side nearly as far from the perpendicular, as the place whence it began to descend; after which it will again descend towards the perpendicular, and thus it will keep moving backwards and forwards for a considerable time; and it would continue to move in that manner for ever, were it not for the resistance of the air, and the friction at the point of suspension, which always prevent its ascending to the same height as that from which it lastly began to descend.*

Thus

Thus the pendulum, fig. 9, Plate IV. being moved from the perpendicular direction CB to the situation AC, and there left to itself, will descend along the arch AB with an accelerated motion, in the same manner as if it descended over a curve surface AB; for it is evidently the same thing whether a body descends along such a surface, or is confined by the string CB, so as to describe the same curve AB. By the time the body arrives at the lowest point B, it will acquire the same velocity as if it had descended perpendicularly from E to B, (by prop. V.) This velocity (if the retardation arising from the resistance of the air and the friction be removed) will carry it beyond the point B with a retarded motion in an equal portion of time, as far as D (see page 71) viz. as far from B as A is from B. It will then descend again with an accelerated motion towards B, and so on. For since the velocity of the pendulum in its ascent is retarded by the same uniformly acting power, which accelerates it in its descent, namely, by the force of gravity, there must be the same time employed in destroying as in generating any momentum.

It likewise follows from this consideration, that the weight of the pendulum cannot alter its time of descent or ascent; for it has been shewn above, that bodies of different weights will move through equal spaces in equal times, towards a centre of attraction, provided the attractive force be the same. And
the

the motion of a pendulum is evidently owing to the gravitating power.

The whole motion of the pendulum one way is called a *vibration* or *oscillation*. Thus the motion of the pendulum from A to D is one vibration; from D to A is another vibration, and so on. The body which hangs by the string, is commonly called the *bob* of the pendulum.

This property of the pendulum is fully confirmed by a variety of experiments. A pendulum, if once moved out of its perpendicular situation, and then left to itself, will move forwards and backwards for a considerable time (in some cases, for many hours); but every vibration will be a little shorter than the preceding, until at last the pendulum will entirely cease to move. That this gradual retardation is entirely owing to the resistance of the air, and to the friction at the point of suspension, is proved by observing that the same pendulum has been found to continue its vibrations longer and longer, in proportion as those causes of obstruction have been diminished; hence we conclude, that if those causes could be entirely removed, the pendulum would continue to vibrate for ever.

Prop. IX. *The velocity of a pendulum in its lowest point is as the chord of the arch which it has described in its descent.*

Thus if there be two pendulums of equal lengths, as CF and CA, fig. 10, Plate IV. and the former of them descends from F, whilst the latter descends from

from

from A ; then at the lowest point B the velocity of the former will be to the velocity of the latter, as the chord or straight line FB is to the chord or straight line AB, or as the velocities acquired by the perpendicular descents GB EB ; which is an evident application of the propositions V. and VI. of this Chapter.

Prop. X. *The very small vibrations of the same pendulum are performed in times nearly equal; but the vibrations through longer and unequal arches are performed in times sensibly different.*

It is evident (from cor. to prop. III.) that if the pendulous body, instead of vibrating along circular arches, could move along the chords of those arches, the semi-vibrations, whether long or short, would be all performed in equal times ; viz. each in the time that a body would employ in descending perpendicularly along the diameter of the circle, or twice the length of the pendulum. For instance, in fig. 10, Plate IV. the pendulous body would descend from F to B or from A to B along the chords or straight lines FB or AB, exactly in the same time, viz. the time it would employ in the perpendicular descent from H to B ; and since the descent from A to B, or from F to B, is half a vibration, therefore each whole vibration would be performed in twice that time.

But since the body vibrates not along the chords but along the arches, therefore the unequal vibrations cannot be performed in equal times (see prop.

IV.); yet in very small arches the chords are nearly equal to the arches that are subtended by them, (see prop. I. of the note in p. 139.) therefore the vibrations along very small arches, though of unequal lengths, are performed in times nearly equal.

Prop. XI. *As the diameter of a circle is to its circumference, so is the time of a heavy body's descent from rest through half the length of a pendulum to the time of one of the smallest vibrations of that pendulum.*

The demonstration of this proposition depends upon certain difficult mathematical principles; we shall therefore subjoin it by way of a note, for the information of those who are qualified to read it; and shall now proceed to shew the use of this curious proposition by means of examples (1).

This

(1) The analogy which is announced in the above proposition, is deduced from the properties of a curve called the *cycloid*. It is therefore necessary, in the first place, to shew the nature and principal properties of that curve, from which the above mentioned analogy may afterwards be derived.

If a circle, as AB, fig. 11, Plate IV. resting on a right line AL, touches it in a point A; and if this circle be rolled along the said line, until the same point A in the circumference, which first touched the line AL, comes again in contact with it in another point L; or till the circle AB, by rolling along the line AL, has performed a whole revolution; then the point A will, by its two-fold motion, describe the curve ACDIL, which is called a *cycloid*.

The circle ABC is called the *generating circle*, AL is the *base*, and DF, erected perpendicularly in the middle of

This proposition shews the proportionality of four quantities; viz. the diameter of a circle, its circumference, the time which a heavy body employs in falling from rest through a certain space, and

the base, and extended from the base to the curve, is the axis of the cycloid.—ABC, DGF, HIK, represent the generating circle in different situations.

From this generation of the cycloid, the following particles are obviously derived.

I. *The base AL is equal to the circumference of the generating circle.*

II. *The axis DF is equal to the diameter of the generating circle.*

III. *The part KL of the base, viz. the part between one extremity of it and the place which touches the generating circle in any situation of it, is equal to the corresponding arch IK, or GF, of the generating circle; the ordinate IE being parallel to the base.*

IV. *FK, or its equal ME, is equal to the remaining arch IH, or GD.*

V. *The chord IK is perpendicular to the curve at I.*

VI. *The chord IH, being perpendicular to IK, (for the angle HIK in the semicircle is a right angle) is a tangent to the curve at the point I.*

VII. *The tangent IH of the curve at I, or chord of the circular arch HI, is equal and parallel to the chord DG. Also IK is equal and parallel to FG.*

VIII. *The length of the semicycloid DIL is equal to twice the diameter DF of the generating circle; and any cycloidal arch ID, cut off by a line IE parallel to the base, is equal to*

and the time of a small oscillation of a pendulum, whose length is equal to twice that space.

It is very well known that the diameter of a circle is to its circumference, as one is to 3,1415 nearly;

twice the chord DG of the corresponding circular arch DG, which is cut off by the same line IE.

Draw PT indefinitely near and parallel to IE, which will cut the circle DGF in Q. Join DQ produce DG to meet TP in S; from Q draw QO perpendicular to DS, and draw GR, a tangent to the circle at G, and RD a tangent at D. Then, since PT is indefinitely near to EI; GS is equal to the increment IT of the curve, whilst GO is the increment of the chord DG; for DQ being nearly equal to DO, must exceed DG by the increment, or additional part GO. And this increment or addition to the chord has been made at the same time that the curve DI has been increased of the part IT, equal to GS.

Now the triangles DRG, GQS, being similar (since DR is parallel to QS, and the angles at the vertex G are equal), and DR being equal to RG, QS must be equal to QG; hence GO is likewise equal to OS, and of course GS is equal to twice GO; but GS is equal to the increment of the curve, and GO is equal to the contemporaneous increment of the chord DG; therefore the increment of the curve is equal to twice the increment of the chord. And as this reasoning is applicable to any point of the curve from D to L, therefore we conclude, that since from the upper point D to the lowest L, the curve increases twice as fast as the corresponding chord of the circle DGF, therefore any arch DI of the curve is equal to twice the corresponding chord DG; and at L where the corresponding
chord

nearly; therefore if one of the other particulars be known, we may find out the fourth by means of the common rule of three.

Example 1st. The time in which a body will descend from rest through 16,087 feet, viz. (one second)

chord is DF, the curve or semicycloid DIL is equal to twice DF, viz. twice the diameter of the generating circle.

IX. *CA, GB, fig. 12, Plate IV. represent two equal semicycloidal cheeks set contiguous to each other with their bases CE, CK, in the same direction. BDA is an inverted cycloid equal to the cycloid of which CA or CB is the half, and its base reaches from the vertex B of one semicycloid to the vertex A of the other. At C suspend a pendulum CLI, whose length is equal to one of the semicycloids. As this pendulum vibrates in the plane of the cycloids, its string will apply itself first to one and then to the other of those cheeks, by which means the end I of the pendulum will move precisely in the curve BDA; viz. in a cycloid.*

It is evident from the construction, that BA is the base of the cycloid BDA; that $BF = FA = CE = CK$, and that $CD = CL = CLB = BID =$ twice the diameter of the generating circle FGD, or EHB.

In any situation of the pendulum, as CLI draw LH through the point where the contact between the string of the pendulum and the cycloidal cheek terminates, and draw IY through the end of the pendulum, both parallel to the base BA; and join the points B, H, G, F, with the lines BH, and GF.

Since CLB is equal to CL the disengaged part LI of the string must be equal to LB, and of course equal to

(second) being given, to find the time in which a pendulum of twice that length (viz. of 32,174 feet) will perform one of its least vibrations.

Here we have $1 : 3,1415 :: 1''$: to a fourth proportional, viz. to $3'',1415$, which is the time in which

twice the chord HB (by § VIII). But BH is equal and parallel to the tangent LM (by §. VII); therefore HB is equal to ML, and consequently LM is equal to MI; hence the parallels HL, IG are equidistant from the base BA, and cut off equal arches HZB, FSG, from the generating semicircles; therefore the chord FG is equal and parallel to the chord HB, and to MI. Also MF is equal to IG, those lines being the opposite sides of a parallelogram. Now as BM is equal to HL, and (by § IV.) equal to the arch HZB, or to the arch FSG; the remainder MF, equal to IG, will be equal to the remaining part GD of the semicircle; which proves that the extremity I of the pendulum is always in the cycloidal curve ADB.

For the sake of brevity we shall call the pendulum which vibrates in a cycloid, a *cycloidal pendulum*.

X. *The velocity of a cycloidal pendulum in its lowest point is proportional to the space passed through; viz. to the arch of the cycloid which the pendulum has described in its descent.*

Thus in fig. 12, Plate IV. If the pendulum begin to descend from I; at D, its velocity will be as the arch ID; and if it begin to descend from B, then when it arrives at the lowest point D, its velocity will be as the arch BID; which we are now going to prove.

It has been shewn in chap. X. prop V. that a body will acquire the same velocity whether it descends obliquely from

which the pendulum of 32,174 feet will perform each of its very small vibrations; viz. little more than three seconds.

Example 2. The time in which a body will descend from rest through 16,087 feet (viz. one second) being given, to find the time in which a pendulum of four feet will perform one of its least vibrations.

Here

from I to D, or perpendicularly from Y to D. Also the square of the velocity of a falling body is as the space passed through, or the velocity is as the square root of the space; therefore the velocity acquired by the pendulum I, in its descent from I to D, is as the square root of YD, viz. as \sqrt{YD} . But (Eucl. p. 8. B. VI.) $DY : DG :: DG : FD$; therefore $DY = \frac{DG \times DG}{FD}$. Now FD

being an invariable quantity, DY must increase or decrease according as the square of DG increases or decreases; or the square root of DY (viz. the velocity in question) is as DG, which is equal to half the cycloidal arch DI; hence the velocity is as the cycloidal arch.

XI. *All the vibrations of a cycloidal pendulum, whether long or short, are performed in equal times.*

In all sorts of motion, as we have abundantly shewn, the space is as the product of the time multiplied by the velocity; viz. S is as TV, which gives the following analogy, $S : V :: T : 1$. But it has been just shewn, that in the case of a cycloidal vibration, the space is as the velocity; therefore the time must be as unity, or always the same.

Here half the length of the pendulum is 2 feet; therefore in the first place we must find out what time a body will employ in descending from rest through 2 feet; and since the spaces passed over by descending bodies, are as the squares of the times, (see page 65) we say as 16,087 feet are to two feet, so is the square of one second to the square of the time sought; viz. $16,087 : 2 :: 1'' : \left(\frac{1 \times 2}{16,087} = \right) 0,1243$, the square root of which, viz. $0'',352$, is the time of a body's descent through 2 feet.

XII. If a cycloidal pendulum begin to descend from any point *L*, fig. 13, Plate IV. towards the vertex *V*; its velocity at any point *M* (viz. the velocity acquired by descending from *L* to *M*) will be as the square root of the difference of the squares of the two arches *VL*, and *VM* (viz. as $\sqrt{VL^2 - VM^2}^{\frac{1}{2}}$); or it will be as the sine of a circular arch whose radius is equal to *VL*, and whose cosine is equal to *VM*.

Through the points *L* and *M* draw *LR*, *MS*, parallel to the base *AB*, which lines will cut the generating circle in *O* and *Q*. And draw the chords *VO*, *VQ*.

The velocity of the pendulum at the point *M*, after a descent from *L*, is equal to the velocity that would be acquired by a body descending perpendicularly from *R* to *S* (by prop. V. of this chap.); and this velocity is as the square root of the space *RS*; or as $\sqrt{RV - SV}^{\frac{1}{2}}$; or as $\sqrt{VO^2 - VQ^2}^{\frac{1}{2}}$; or, lastly, as $\sqrt{VL^2 - VM^2}^{\frac{1}{2}}$. (See the demonstration of the proposition last but one.)

Produce

2 feet. This time being found, we then say, after the manner of the preceding example; $1 : 3,1415 : : 0'',352 : \left(\frac{3,1415 \times 0'',352}{1} = \right) 1'',1$ nearly the time in which a pendulum of 4 feet performs its least vibrations.

Example 3. The time in which a pendulum of 39,1196 inches performs each of its small vibrations (*viz.* one second) being given, to find the space through which a body will descend from rest in the same time.

First

Produce the axis DV towards Z; at V erect VL perpendicular to DZ, and equal to the length of the cycloidal arch VML. Let the lengths VM, VL, in the straight line VL, be made respectively equal to the lengths VM, VL, of the cycloidal arch. With the centre V and radius VL, draw the semicircle LXP. At M on the radius erect MX perpendicular to it, which will meet the circumference at X, and lastly join VX.

Then MX is the sine of a circular arch, whose radius is VX or VL, which is equal to the cycloidal arch VL, and whose cosine is VM, which is equal to the cycloidal arch VM. (By Eucl. p. 47. B.1.) MX is equal to $\sqrt{VX^2 - VM^2}^{\frac{1}{2}}$, or to $\sqrt{VL^2 - VM^2}^{\frac{1}{2}}$.

XIII. If when the pendulum begins to descend from L along the cycloid, another body be supposed to move in the semicircle LXP from L towards Z with a uniform velocity, equal to the pendulum's greatest velocity; (*viz.* that which the pendulum acquires by descending from L to the vertex V;) then

First we say $3,1415 : 1 :: 1'' : 0'',3183$, the time in which a body will descend through a space equal to half the length of the pendulum, viz. through 19,5598 inches.

Then,

then any circular arch XY will be described by the above-mentioned body with that uniform velocity, in the same time that the cycloidal arch which is intercepted between the two corresponding points M and N , is run over by the pendulum with its usual accelerated velocity.

Draw the line mx parallel, and indefinitely near, to the sine MX . Through X draw Xr parallel to the radius VL ; and in the cycloidal arch take Mm equal to Mm in the radius.

The arch Xx , being indefinitely small, may be considered as a right line. Then the right angled triangles VMX , Xxr , being similar, (because the angles rXx and MXV are equal, for each of them is the complement of VXr to a right angle), we have $MX : VX$ (or VZ , or VL) :: Xr (or Mm) : Xx .

Now the velocity of the pendulum at M (by § XII. of this note) is as MX ; therefore the extremely small space Mm in the arch, may without error be supposed to be described with that velocity. Also (by § X. of this note) the greatest velocity acquired by the pendulum in its descent from L to V , is as the arch LV , or as its equal, the radius LV , and is the same velocity with which the circular arch is equably described; therefore, the analogy of the preceding paragraph is, by substitution, converted into the following: viz. as the velocity with which the circular arch Xx is described, is to the velocity with which the small cycloidal arch Mm is described by the pendulum, as Xx is

Then, since the spaces described by descending bodies, are as the squares of the times, we say, as the square of 0'',3183 is to the square of one second, so are 19,5598 inches to the space through which a body will descend in one second; viz. 0,10131489 : 1 :: 19,5598 : 193,06 inches, or 16,083 feet; the space through which a body will descend from rest in one second.

In

to Mm ; so that those small lines are as the velocities with which they are described. But when the spaces are as the velocities, the times must be equal; therefore, the circular arch Xx is described in the same time that the corresponding cycloidal arch Mm is described by the pendulum. Now as the same thing may be said of all other corresponding parts between X and Y , and M and N ; therefore the whole circular arch XY is described in the same time in which the corresponding cycloidal arch MN is described. Hence the whole cycloidal arch LV , and quadrant LZ , are described in the same time.

XIV. The time of a complete oscillation of a cycloidal pendulum, is to the time in which a body would descend perpendicularly along the axis of the same cycloid, as the circumference of a circle is to its diameter.

In the first place, it is evident that the time in which the semicircle LZP is described in the manner mentioned above, is to the time in which the radius LV could be described with the same equable velocity, as the circumference of a circle is to its diameter. But the time in which the semicircle LZP is described, is equal to the time in which the pendulum will make a complete cycloidal oscillation from L to

In the preceding examples the calculations have not been carried on to a great number of decimals, purposely to avoid prolixity; the object being only to shew the method of performing the calculations; but in many cases it will be necessary to extend the operation to a greater degree of accuracy. It is likewise necessary that the reader be informed of the real length of the pendulum, which vibrates
seconds,

to P. And the time in which LV (or its equal twice OV) could be described with that same velocity with which the circle is described, is equal to the time of descent along the chord OV, or along the axis DV (see chap. V. and prop. VI. of this chap.); therefore, the above-mentioned analogy is, by substitution, converted into the following. The time of a complete cycloidal oscillation, is to the time in which a body would descend perpendicularly along the axis of the same cycloid, as the circumference of a circle is to its diameter.

This proposition evidently confirms prop. the 11th of this note; for since every cycloidal vibration is in the same ratio to the time of descent through the axis, as the invariable ratio of the circumference of a circle to its diameter, they must be all performed in the same time.

The very small vibrations of a common circular pendulum, may without any sensible error be supposed to follow the same laws as those of the cycloidal pendulum; for near the vertex V, that is, when the arch of vibration does not exceed two or three degrees, the curvature of the cycloid coincides with the curvature of a circle whose radius is equal to CV. viz. the length of the pendulum. This is
evidently

seconds, or of the real space which is passed over by descending bodies in a given time; (since the one may be easily deduced from the other) in order that he may ground his calculations on as accurate a foundation, as the present state of knowledge can admit of.

In different parts of the world, the pendulum which vibrates seconds, is not of the same length. It

evidently shewn by the figure itself; for when the pendulum vibrates not far from the perpendicular CV, its string does hardly touch the cycloidal cheeks CA, CB; and of course its extremity V must describe a circular arch very nearly.

XV. The times of similar oscillations of different pendulums, when the force of gravity is supposed to vary, are as the square roots of the lengths of the respective pendulums directly, and as the square roots of the respective gravitating forces inversely.

By the preceding proposition the time of a cycloidal vibration is to the time of perpendicular descent along the axis, in an invariable ratio; that is, the former is as the latter. Now the time of that perpendicular descent is directly as the square root of the axis (or of its double, viz. the length of the pendulum), and inversely as the square root of the force of gravity; for when the gravitating force is invariable, the time of perpendicular descent has been shewn to be as the square root of the space; and when the time is invariable, (viz. in the same time) the square root of the space has been shewn to be as the square root of the velocity, or of the gravitating force; therefore when they are both variable, the square root of the space or length is as the time multiplied

It is a little longer on places that are situated nearer to the poles, and shorter in situations that are nearer to the equator, (the reason of which will be shewn hereafter). This difference, however, is known with sufficient accuracy. But most philosophical writers differ with respect to the length of the pendulum which vibrates seconds in the same latitude; and of course with respect to the real

multiplied by the square root of the gravitating force; or the time is as the square root of the space divided by the square root of the gravitating force; that is, as the square root of the length of the pendulum directly, and the square root of the force of gravity inversely.

Independently of the cycloid, the time of any circular oscillation may be found out by means of the following problem, which is given by Professor Saunderson in his Method of Fluxions.

XVI. *To find the exact time of one of the least oscillations of a given pendulum swinging in an arch of a circle; and to find, without any sensible error, also the time of any other oscillation.*

Let a pendulum ND, fig. 14, Plate IV. vibrate in the arch ADC of a circle whose diameter is ID; and suppose it to be at the point E in its ascent from D to C. Let \sqrt{BF} express the velocity acquired by a heavy body in falling from B to F, (AC, EE, being the parallel chords of the arches ADC, EDE, which intersect the diameter in B and F), and consequently the velocity of the pendulum at the point E. Now $\frac{1}{2} \sqrt{ID}$ expresses the velocity acquired by descending through $\frac{1}{4} ID$, and since a body with that velocity would

real space which is passed over by descending bodies in a given time.—In the works of the most eminent philosophers of this country, I find the length of the pendulum, which vibrates seconds in or near London, stated differently as follows: inches 39,2; 39,14; 39,128; 39,125; 39,1196, &c.

The

would describe uniformly a space equal to $\frac{1}{2}$ ID in the same time in which it would fall through $\frac{1}{4}$ ID. Divide the space $\frac{1}{2}$ ID by the velocity $\frac{1}{2} \sqrt{ID}$, and the quotient \sqrt{ID} , expresses the time wherein a heavy body would fall through a $\frac{1}{4}$ ID; viz. through half the length of the pendulum.

Draw ee indefinitely near to EE ; then Ee may be considered as the fluxion of the arch DE ; and $\frac{Ee}{\sqrt{BF}}$ will ex-

press the time wherein the small arch Ee is described by the pendulum, or the fluxion of the time of a vibration. But

$Ee = \frac{\frac{1}{2}ID \times Ff}{\sqrt{IF \times FD}}$ (for, calling the radius ND , r ; FE , y ; and

FD , x ; we shall have $Ee = \sqrt{\dot{x}^2 + \dot{y}^2}^{\frac{1}{2}}$. But $y^2 = 2rx - xx$,

whose fluxion is $2y\dot{y} = 2r\dot{x} - 2x\dot{x}$; hence $\dot{y} = \frac{r\dot{x} - x\dot{x}}{y}$; or

$\dot{y}^2 = \frac{r^2\dot{x}^2 - 2rx\dot{x}^2 + x^2\dot{x}^2}{y^2} = \frac{r^2\dot{x}^2 - y^2\dot{x}^2}{y^2} = \frac{r^2\dot{x}^2}{y^2} - \dot{x}^2$. There-

fore $\dot{x}^2 + \dot{y}^2 = \frac{r^2\dot{x}^2}{y^2}$, and $\sqrt{\dot{x}^2 + \dot{y}^2}^{\frac{1}{2}} = \frac{r\dot{x}}{y} = \frac{r\dot{x}}{\sqrt{2rx - xx}} =$

$\frac{\frac{1}{2} ID \times Ff}{\sqrt{IF \times FD}} = Ee \Big) = \frac{\sqrt{ID}}{\sqrt{IF}} \times \sqrt{ID} \times \frac{\frac{1}{2} Ff}{\sqrt{FD}}$; there-

fore, $\frac{Ee}{\sqrt{BF}} = \frac{\sqrt{ID}}{\sqrt{IF}} \times \sqrt{ID} \times \frac{\frac{1}{2} Ff}{\sqrt{BF \times FD}}$. Bisection DB

in

The late Mr. John Whitehurst, an ingenious member of the Royal Society, seems to have contrived

in K, and KD in L; and when the arch ADC is small, the quantity IF cannot differ sensibly from IK, nor $\frac{\sqrt{ID}}{\sqrt{IF}}$ from $\frac{IL}{IK}$. Therefore $\frac{Ee}{\sqrt{BF}}$ is very nearly equal to $\frac{IL}{IK} \times \sqrt{ID} \times \frac{\frac{1}{2} Ff}{\sqrt{BF \times FD}}$.

Upon the diameter BD describe the circle BGDG, cutting the chords EE, ee in G and g; then will the fluxion of the arch DG be $Gg = \frac{\frac{1}{2} BD \times Ff}{\sqrt{FB \times FD}}$; consequently $\frac{Gg}{BD} = \frac{\frac{1}{2} Ff}{\sqrt{FB \times FD}}$; and therefore the fluxion of the time of vibration through DE will be $\frac{Ee}{\sqrt{BF}} = \frac{IL}{IK} \times \sqrt{ID} \times \frac{Gg}{BD}$; which in fact is the time of the pendulum's moving from E to e.

But the fluent of this last fluxion is $\frac{IL}{IK} \times \sqrt{ID} \times \frac{DGB}{BD}$; this, therefore, is the time of half a vibration or motion of the pendulum from D to C. And the time of a whole vibration through the arch ADC is $\frac{IL}{IK} \times \sqrt{ID} \times \frac{BGDGB}{BD}$.

When

trived and performed the least exceptionable experiments relatively to this subject. The result of his

When the arch of vibration is indefinitely small, the quantity $\frac{IL}{IK}$ becomes $= 1$; and the time (T) of one of the least vibrations, will become $T = \sqrt{ID} \times \frac{BGDGB}{BD}$; and therefore $BD : BGDGB :: \sqrt{ID} : T$; that is, as the diameter of a circle is to its circumference, so is the time (\sqrt{ID}) of the descent through half the length of the pendulum, to the time of one of the least oscillations of the pendulum; which is the same analogy as was derived from the properties of the cycloid. Wherefore the time of oscillation in a cycloid, and in an indefinitely small arch of a circle, is the same, viz. $T = 1$ second, when the length of the pendulum is 39,1196 inches; as has been proved experimentally.

Therefore, the time of an oscillation in a circular arch in general, is $T \times \frac{IL}{IK}$, or (since $IL = IK + KL$) the general expression of the time of vibration through any arch ADC of a circle will be $T + T \times \frac{KL}{IK}$. And $T \times \frac{KL}{IK}$ is the excess of the time of vibration in a circular arch, above the time of vibration in the arch of a cycloid, or above the time of the least circular oscillation; the lengths of the pendulums being equal.

In order to adapt the preceding expressions to the practical calculation, it is necessary to observe that BD is the versed sine of CD, viz. of half the arch of vibration. DK is the half of that versed sine; KL is a quarter of it; and ID is

his experiments shews, that the length of the pendulum which vibrates seconds in London, at 113 feet

twice the length of the pendulum ; hence if we call the versed sine of half the arch of vibration a , and call the length of the pendulum b ; then the above stated expression

$$T + T \times \frac{KL}{IK} \text{ will become } T + T \times \frac{\frac{1}{4}a}{2b - \frac{1}{2}a}, \text{ or } T + T \times \frac{a}{8b - 2a}.$$

Example 1. Suppose it be required to find the time in which a pendulum, that performs each of its smallest vibrations in one second, will perform its vibrations in an arch of 120° .

In this case the length of the pendulum is $b = 39,196$; the femiarch of vibration is 60° ; and its versed sine (which is taken from the trigonometrical tables, and is reduced in the proportion of the tabular radius to the length of the pendulum ; by saying, as the tabular radius is to the length of the pendulum, so is the tabular versed sine to the versed sine in question) is $19,5598 = a$; therefore the time sought is

$$T + T \times \frac{a}{8b - 2a} = 1'' + 1'' \times \frac{19,5598}{273,8372} = 1'',0714 ;$$

then if the number of seconds in 24 hours, viz. $86400''$ be divided by the time of one vibration last found ; viz. by $1'',0714$, the quotient 80735 is the number of vibrations which the pendulum will perform in 24 hours, when it vibrates along the arch of 120° ; whereas when the same pendulum performs very small vibrations, it will vibrate exactly seconds, viz. it will perform 86400 vibrations in 24 hours.

Example 2. Suppose it be required to find the time of one vibration, when the above-mentioned pendulum vibrates through

feet above the level of the sea, in the temperature of 60° of Fahrenheit's thermometer, and when the barometer

through a semicircle. In this case the versed sine is equal to the radius, or to the length of the pendulum, viz. $a = b$; consequently the expression $T + T \times \frac{a}{8b - 2a}$ becomes

$1'' + 1'' \times \frac{a}{8b - 2b} = 1'' + \frac{1''}{6} = 1'', 166666$; so that the time of a small vibration of the pendulum, whose length is 39,196 inches, is to the time of one of its vibrations along a semicircle as 1 is to 1,16666, which is nearly in the proportion of 6 to 7.

We shall conclude this long note with the demonstration of another curious property of the cycloid.

XVII. *If two points be given in a vertical plane, but not both in the same line perpendicular to the horizon, a body will descend from the upper point to the lower in the shortest time possible, if it be caused to move along the arch of a cycloid, which passes through those points, and whose base is an horizontal line that passes through the upper point.*

Thus if the two points be A and B, fig. 15, Plate IV. and it be required that a body should descend from A to B in the shortest time possible; this object will be obtained by causing the body to descend not along the straight line AB, as it might at first sight be imagined, nor along an arch of a circle, or other curve; but along the cycloid ADB, which passes through the given points A and B, and whose base is the horizontal line AO.—On account of this remarkable property, the cycloid is called *the line of swiftest descent*.—We shall divide the demonstration of this property into three parts.

barometer is at 30 inches, is 39,1196 inches; whence it follows that the space which is passed over by bodies

1. If a certain line, as ACDB, be the line of swiftest descent between two points A and B; it follows that a body, after its descent from A as far as C, will continue to descend quicker along the same line from C to D, than along any other line, as CED; for if this be denied, then it must be admitted that the body will descend faster along the line ACEDB, than along the line ACFDB; consequently the line ACFDB is not the line of swiftest descent, which is contrary to the hypothesis.

2. Let ADGB, fig. 16, Plate IV. be a curve between the two given points A and B; let DE, EG, be two indefinitely small and contiguous portions of it. Through the points D, E, and G, draw DL, EO, GP, perpendicular to the base AC; and through D draw DH parallel to the base. Now if this curve be such that the velocity with which the indefinitely small portion DE is passed over by a body after its descent from A to D, be always proportional to $\frac{DH \times a}{DE}$ (a being a certain invariable line or quantity); then the body after its descent from A to D, will descend along the curve from D to G in less time than along any other way DFG; and of course this curve will be the line of swiftest descent.

Through F draw FQ parallel to EG, and let FQ be supposed to be passed over with the same velocity as EG; draw FN perpendicular to DE, as also ME and GQ perpendicular to FQ, then the triangle FNE being similar to DEH, as also FME similar to GEI, we have $DE : DH :: FE : NE$ $= \frac{DH \times FE}{DE}$; and $GE : EI :: FE : FM$ $= \frac{EI \times FE}{GE}$.

Hence

bodies descending perpendicularly, in the first second of time, is 16,087 feet.—This length of a
second

$$\text{Hence } NE : FM :: \frac{DH \times FE}{DE} : \frac{EI \times FE}{GE} :: \frac{DH}{DE} :$$

$$\frac{EI}{GE} :: \frac{DH \times a}{DE} : \frac{EI \times a}{GE}; \text{ viz. } NE \text{ is to } FM \text{ as the}$$

velocity with which NE is passed over to the velocity with which FM is passed over: whence NE, FM, are passed over in equal times. And since MQ is equal to EG, the time of descent through MQ will be equal to the time of descent through EG; so that the time of descent through FQ will be equal to the time of descent through NEG. But since the angle MQG is a right one, FG is greater than FQ; so that the time through FG will be greater than the time through FQ, or through NEG; and since DF is greater than DN, the time through DF will be greater than the time through DN. Whence the time of descent along DF, FG, will be greater than the time of descent along DN, NG. A heavy body therefore, after its fall from A to D, will descend from D to G along the curve DEG, in less time than along any other line; consequently the curve ADEGB is the line of swiftest descent between the points A and B.

3. Let ADEM, fig. 17, Plate IV. be a cycloid whose base is the horizontal line AG. Through any point D in it draw DQ parallel to the base AG, and cutting the generating circle at N and the axis at Q. Draw the chords GN, NM; through D draw DL perpendicular to the base; and draw OE indefinitely near and parallel to LD. Now the indefinitely small part DE of the curve may be considered as a right line coinciding with the tangent at D, and

second pendulum is certainly not mathematically exact, yet it may be considered as such for all common purposes; for it is not likely to differ from the truth by more than $\frac{1}{10000}$ th part of an inch.*

XII. *The*

it may likewise be supposed to be described by a body descending from A, with the same velocity which the body has acquired by its descent from A to D; for the acceleration of velocity through that indefinitely small space may be considered as next to nothing. Now we shall prove that this cycloid has the property of the above-mentioned curve, viz. that the velocity with which the small portion DE is described by a body falling from A, is always proportional to $\frac{DH \times a}{DE}$; (*a* denoting the axis GM of the cycloid).

From the above-mentioned properties of the cycloid, the small line DE, coinciding with the tangent at D, is parallel to the chord NM. Whence the triangles DHE, NQM, and GMN, are equiangular and of course similar; therefore $DE : DH :: GM (=a) : GN = \frac{DH \times a}{DE}$. But GN

is as the velocity which is acquired by the heavy body in its descent from G to Q, or from L to D; viz. as the velocity with which the indefinitely small line DE is passed over; therefore the cycloid, having the property of the above-mentioned curve, is the line of swiftest descent, &c.

* See Mr. Whitehurst's attempt towards obtaining inva-
riable measures of length, capacity, and weight. Also Sir
George Shuckburg Evelyn's excellent paper on the stand-
ard of weight and measure, in the Philosophical Transactions
for the year 1798.

XII. *The times in which similar vibrations (viz. vibrations through arches of the same number of degrees) of different pendulums are performed, are as the square roots of the lengths of the pendulums.*

Thus if the pendulum AB, fig. 18, Plate IV. be four times as long as the pendulum CD, then the time of a vibration of the former will be double the time of a similar vibration of the latter. For (by cor. to prop. VII. of this chap.) the vibrations, and of course the semivibrations, being similar and similarly situated, the time of the pendulum's descent along the arch GB is to the time of the other pendulum's descent along the arch HD, as the square root of GB is to the square root of HD. But the circumferences of circles, or similar portions of the circumferences, are as their radii; therefore the square roots of similar portions of the circumferences are as the square roots of the radii; consequently the times of similar vibrations are as the square roots of the radii, or of the lengths of the pendulums.

Throughout the present chapter the force of gravity has been supposed invariable; but when that is not the case, as for instance, when a pendulum, which vibrates near the surface of the earth, is compared with a pendulum on the top of a very high mountain, or with a pendulum which vibrates on an inclined plane; in which cases the action of the gravitating force on the pendulum's is not the same, then *the time of vibration is as the quotient of the*

200 *Of the Centre of Oscillation, and
square root of the length of the pendulum divided by
the square root of the gravitating force.*

This proposition will be found demonstrated in
the note.

CHAPTER XI.

OF THE CENTRE OF OSCILLATION, AND CENTRE OF PERCUSSION.

THE attentive reader must undoubtedly have remarked, that though in the preceding chapter much has been said with respect to the length of the pendulum, yet no mention has been made of the point from which that length, or distance from the point of suspension, should be measured. The reason of this omission is, that the determination of that point, which is called *the centre of oscillation*, requires a very particular consideration; such indeed as could not without obscurity be introduced in the preceding chapter. We shall now endeavour to elucidate the nature of that point, and to lay down the methods of determining its situation or distance from the point of suspension in pendulums of different lengths and shapes.

When

When the pendulum consists of a spherical body fastened to a string, a person unacquainted with the subject might at first sight imagine that the length of the pendulum must be estimated from the point of suspension to the centre of the ball. But this is not the case; for in fact the real length of the pendulum is greater than that distance, the reason of which is, that the spherical body does not move in a straight line, but it moves in a circular arch; in consequence of which, that half of it which is farthest from the point of suspension, runs through a longer space than the other half which is nearer to the point of suspension; hence the two halves of the ball, though containing equal quantities of matter, do actually move with different velocities, therefore their momentums are not equal; and it is in consequence of this inequality that the centre of oscillation does not lie between the two hemispheres; that is, in the centre of the ball; but it lies within the lower hemisphere, viz. that which has the greater momentum. Now from this it naturally follows, that if the ball of the pendulum could be concentrated in one point, that point would be the centre of oscillation; so that the centre of oscillation is that point wherein all the matter (and of course the forces of all the particles) of the body or bodies that may be joined together to form a pendulum, may be conceived to be condensed.

The *centre of percussion* is that part or point of a pendulous body, which will make the greatest impression

pression on an obstacle that may be opposed to it whilst vibrating; for if the obstacle be opposed to it at different distances from the point of suspension, the stroke, or percussion, will not be equally powerful; and it will soon appear that this centre of percussion does not coincide with the centre of gravity.

Let the body AB, fig. 1, Plate V. N. I, consisting of two equal balls fastened to a stiff rod, move in a direction parallel to itself, and it is evident that the two balls must have equal momentums, since their quantities of matter are equal, and they move with equal velocities. Now if in its way, as at N. II, an obstacle C be opposed exactly against its middle E, the body will thereby be effectually stopped, nor can either end of it move forwards, for they exactly balance each other, the middle of this body being its centre of gravity. Now should an obstacle be opposed to this body, not against its middle, but nearer to one end, as at N. III, then the stroke being not in the direction of the centre of gravity, is in fact an oblique stroke, in which case, agreeably to the laws of congress which have been delivered in chap. VIII. a part only of the momentum will be spent upon the obstacle, and the body advancing the end A, which is farthest from the obstacle, as shewn by the dotted representation, will proceed with that part of the momentum which has not been spent upon the obstacle; consequently in this case the percussion is not so powerful as in the foregoing. Therefore there is a certain point in a
moving

moving body which makes a stronger impression on an obstacle than any other part of it.—In the present case, indeed, this point coincides with the centre of gravity; because the two ends of the body before the stroke moved with equal velocities. But in a pendulum the case is different; for let the same body of fig. 1, Plate V. be suspended by the addition of a line AS, fig. 2, Plate V. which line we shall suppose to be void of weight and flexibility, and let it vibrate round the point of suspension S. It is evident that now the two balls will not move with equal velocities; for the ball B, by describing a longer arch than the ball A in the same time, will have a greater momentum; and of course the point where the forces of the two balls balance each other, which is the centre of percussion, lies nearer to the lower ball B; consequently this point does not coincide with the centre of gravity of the body AB; but it is that point wherein the forces of all the parts of the body may be conceived to be concentrated. Hence the centre of oscillation and the centre of percussion coincide; or rather they are exactly the same point, whose two names only allude, the former to the time of vibration, and the latter to its striking force.

If in fig. 1, Plate V. the balls A and B be not equal, their common centre of gravity will not be in the middle at E, but it will lie nearer to the heavier body, as at D, supposing B to be the heavier body; so that the distances BD, AD, may be inversely

versely as the weights of those bodies. Now when the above-mentioned body is formed into a pendulum, as in fig. 2, though the weights A and B be equal, yet by their moving in different arches, viz. with different velocities, their forces or momentums become actually unequal; therefore in order to find the point where the forces balance each other, so that when an obstacle is opposed to that point, the moving pendulum may be effectually stopped, and no part of it may preponderate, in which case the obstacle will receive the greatest impression; we must find first the momentums of the two bodies A and B, then the distances of those bodies from the centre of percussio, or of equal forces, must be inversely as those momentums. Thus the velocities of A and B are represented by the similar arches which they describe, and those arches are as the radii SA, SB. Therefore the momentum of A is the product of its quantity of matter multiplied by SA, and the momentum of B is the product of its quantity of matter multiplied by SB; consequently AD must be to BD, as the weight of B multiplied by SB is to the weight of A multiplied by AS. Then D is the centre of percussio. And since, when four quantities are proportional, the product of the two extremes is equal to the product of the two means; therefore if the weight of A multiplied by AS, be again multiplied by AD, the product must be equal to the product of the weight of B multiplied by BS, and again multiplied

by

by BD; that is, *the product of the body on one side of the centre of oscillation multiplied by both its distance from the point of suspension and its distance from the centre of oscillation, is equal to the product of the body on the other side of the centre of oscillation, multiplied both by its distance from the point of suspension, and its distance from the centre of oscillation.*

The same reasoning may evidently be applied to a pendulum consisting of more than two bodies connected together, or to the different parts of the same pendulous body; hence we form the following general law.

If the weight of each part of a simple or compound pendulum be multiplied both by its distance from the centre of suspension, and its distance from the centre of oscillation or percussion, the sums of the products, on each side of the centre of oscillation, will be equal to each other.

From this law the rule for determining the distance of the centre of oscillation from the point of suspension is easily deduced; but the application of it is attended with considerable difficulty, on which account we shall subjoin it in the note (1), and shall now proceed to shew an experimental or mechanical

(1) Let a pendulum consist of any number of parts or small bodies A, B, C, D, E, joined together; let a, b, c, d, e , stand for their respective distances from the point of suspension; and x for the distance of the centre of oscillation from the point of suspension.

The

chanical method of finding the centre of oscillation; which method is general and easy, at the same time that it admits of sufficient accuracy.

The

The distances of those parts, or bodies, from the centre of oscillation will be $x - a$, $x - b$, $x - c$, $d - x$, $e - x$; D and E being supposed to be farther from the point of suspension, than the centre of oscillation is. By multiplying every one of those bodies, both by its distance from the centre of suspension and its distance from the centre of oscillation, we have, agreeably to the above-mentioned law, the equation $Aax - Aaa + Bbx - Bbb + Ccx - Ccc = Ddd - Ddx + Eee - Eex$; which, by transposition and division, is resolved into the following; viz.

$$x = \frac{Aaa + Bbb + Ccc + Ddd + Eee}{Aa + Bb + Cc + Dd + Ee}.$$

Should any of the bodies, as for instance A and B, in the preceding instance, be situated above the centre of suspension, then their distances will be negative, viz. $-a$, $-b$, though their squares aa , bb , are always positive. In this case the value of x is $= \frac{Aaa + Bbb + Ccc + Ddd + Eee}{-Aa - Bb + Cc + Dd + Ee}$.

Since the centre of gravity of a body or system of bodies, is that point wherein all their matter may be conceived to be condensed, therefore the product of all the matter or sum of the different weights A, B, C, D, E, multiplied by the distance of the common centre of gravity from the point of suspension, is equal to the sum of the products of each body multiplied by its distance from the point of suspension. Hence the above stated value of x becomes $\frac{Aaa + Bbb + Ccc + Ddd + Eee}{Aa + Bb + Cc + Dd + Ee}$ divided by the product of the whole body or sum of the weights, multiplied by the distance of the

The body whose centre of oscillation, or (which is the same) of percussion, is to be ascertained, must be suspended to a pin or other support, but as freely

the centre of gravity from the point of suspension. And being expressed entirely in words, it forms the following general

Rule 1. If all the bodies or parts of a body, that forms a pendulum, be multiplied each by the square of its distance from the point or axis of suspension, and the sum of the products be divided by the product of the whole weight of the pendulum, multiplied by the distance of the centre of gravity from the point of suspension; the quotient will be the distance of the centre of oscillation or percussion from the point of suspension.

The situation of the centre of oscillation may also be found by means of another rule, which we shall likewise lay down, and shall demonstrate; since in some cases this rule will be found preferable to the first.

Rule 2. If the sum of the products of all the parts or weights, multiplied each by the square of its distance from the centre of gravity, or from a line passing through the centre of gravity parallel to the axis of vibration, be divided by the product of the whole mass or body, multiplied by the distance of the centre of gravity from the point of suspension, the quotient will be the distance of the centre of oscillation from the centre of gravity; which being added to the distance of the centre of gravity from the point of suspension, will be the distance of the centre of oscillation from the point of suspension.

Let C A B fig. 4, Plate V. represent any sort of body regular or irregular, suspended at C; O its centre of oscillation; G its centre of gravity; C O B its axis or right line passing through the point of suspension, and centres of gravity

freely as may be practicable; and being once moved out of the perpendicular situation, must be suffered to perform very short vibrations; viz. so small as to be just discernable. Then by keeping an eye on

vity and oscillation. This body may be conceived to consist of an indefinite number of extremely small parts or weights. Let W be one of those small weights; join WC and WG , and from W drop WF perpendicular to CO . Then the product of W , by the square of its distance from C , is $W \times \overline{CW}^2$. But (Eucl. p. 47. B. I.) $\overline{CW}^2 = \overline{WF}^2 + \overline{CF}^2$; and $\overline{GW}^2 = \overline{GF}^2 + \overline{WF}^2$. (Eucl. p. 7. B. II.) $\overline{CG}^2 + \overline{GF}^2 = 2\overline{CG} \times \overline{GF} + \overline{CF}^2$; and by transposition $\overline{CF}^2 = \overline{GF}^2 + \overline{CG}^2 - 2\overline{CG} \times \overline{GF}$. Then by substitution (viz. by putting instead of \overline{CF}^2 , its equal $\overline{GF}^2 + \overline{CG}^2 - 2\overline{CG} \times \overline{GF}$) the above stated equation becomes $\overline{CW}^2 = \overline{WF}^2 + \overline{GF}^2 + \overline{CG}^2 - 2\overline{CG} \times \overline{GF} =$ (putting \overline{GW}^2 for its equal $\overline{GF}^2 + \overline{WF}^2$) $\overline{GW}^2 + \overline{CG}^2 - 2\overline{CG} \times \overline{GF}$. And multiplying both sides by W , we have the sum of all the products $W \times \overline{CW}^2 =$ the sum of all the $W \times \overline{GW}^2 +$ all the $W \times \overline{CG}^2 -$ the sum of all the $W \times 2\overline{CG} \times \overline{GF}$.

But by the nature of the centre of gravity the sum of all the $W \times \overline{GF}$ is $= 0$; for those which are one side of the axis must balance those which are on the other side; and of course all the $W \times 2\overline{CG} \times \overline{GF}$ also become $= 0$. Therefore there remains the sum of all the $W \times \overline{CW}^2 =$ sum of all the $W \times \overline{GW}^2 +$ sum of all the $W \times \overline{CG}^2 =$ sum of all the $W \times \overline{GW}^2 +$ the whole body $\times \overline{CG}^2$. Or (taking away the sum of all the $W \times \overline{GW}^2$ from both

on a clock or watch with a seconds hand, the observer must count the vibrations, and, if possible, even the part of a vibration, that are performed by that

sides of the equation) the sum of all the $W \times \overline{CG}^2 =$ the whole body $\times \overline{CG}^2$.

Then $CO = \frac{\text{sum of all the } W \times \overline{CW}^2}{\text{the whole body} \times CG}$ (by rule the 1st) $= CG + \frac{\text{sum of all the } W \times \overline{GW}^2}{\text{the whole body} \times CG}$. And lastly, $GO = CO - CG = \frac{\text{sum of all the } W \times \overline{GW}^2}{\text{the whole body} \times CG}$; which is rule the 2d.

In the application of the above-mentioned rules, it is frequently very difficult to find the sum of the products of all the weights multiplied by the squares of their respective distances. The method of fluxions is undoubtedly the most extensive, as it may be applied to all such figures or bodies as have some regularity of shape, or such as may be expressed by an algebraical equation. But in some cases the irregularity of form is so very great, that the centre of oscillation can only be found out by means of the above-described mechanical method.

In order to find the sum of the weights, &c. you must consider an indefinitely small part, or increment, or fluxion, of the figure, as being a small weight, and multiply it by the square of its distance from the centre of suspension or axis of vibration, according to rule the 1st, or else multiply it by the square of its distance from the centre of gravity, or from a line passing through the centre of gravity, and parallel to the axis of vibration, according to rule the 2d.; then

that pendulum in one minute, and note the number. N. B. Should the pendulum appear likely to stop before the expiration of the minute, a gentle and

the fluent of that expression will be the sum of the products of all the weights, multiplied by the squares of their respective distances, either from the axis of vibration, or from the centre of gravity, &c. Lastly, this fluent must be divided by the product of the whole body (to be had by common mensuration) multiplied by the distance of the centre of gravity, from the point of suspension; and the quotient will be the distance of the centre of oscillation either from the point of suspension, or from the centre of gravity, according as the operation was performed either by rule the first, or rule the second.

A few examples will render the application of this method more intelligible.

Example 1. Let CB, fig. 5, Plate V. be a right line, or very slender cylinder suspended at C; and call it a , (meaning either its length or weight, for the one is proportionate to the other). G is its centre of gravity. Now if you call any part of this line x , reckoning from C, then the increment or fluxion of x is \dot{x} , which \dot{x} may be considered as one of the vast many weights which form the whole line or slender cylinder. The product of this weight by the square of its distance from C is $x^2\dot{x}$, and the fluent of this expression is $\frac{x^3}{3}$, which, when x represents the whole extent CB, becomes $\frac{a^3}{3}$, and is the sum of the products of all the weights multiplied by the squares of their respective distances from C.

We

and dexterous application of a finger once or twice, will increase a little its vibrations, and prolong its action without altering the time of vibration.—

For

We must now find the product of the whole line multiplied by the distance of the centre of gravity G from C.

But $CG = \frac{a}{2}$, therefore the product in question is $a \times \frac{a}{2}$,

or $\frac{a^2}{2}$. Lastly, divide the above fluent $\frac{a^3}{3}$ by the last pro-

duct, viz. $\frac{a^3}{3} \div \frac{a^2}{2}$, and the quotient $\frac{2a}{3}$ is the distance of the centre of oscillation O from the point of suspension C; so that CO is equal to $\frac{2}{3}$ of CB.

Example 2. Let AB, fig. 6, Plate V. be a right line or very slender cylinder fastened to a line GO void of weight, and suspended at O. The ends A and B are equidistant from O, and the axis of vibration is perpendicular to the plane which passes through ABOG; so that every part of the given line from A to G, or from B to G, is at a different distance from the axis of suspension. Put $OG = a$, and $GP = x$, whose fluxion is \dot{x} , and is a particle or small weight of the given line, which multiplied by the square of OP, which is (Eucl. p. 47. B. 1.) $= \overline{a^2 + x^2}$, becomes $a^2\dot{x} + x^2\dot{x}$. The fluent of this expression is $a^2x + \frac{x^3}{3}$.

The product of the body GP by the distance of the centre of gravity G from O is $(GP \times OG).ax$. Therefore the

distance of the centre of oscillation is $\left(\frac{a^2x + \frac{x^3}{3}}{ax} = \right) a +$

For the sake of greater accuracy, the attentive observer may count the number of vibrations for a longer time, as, for instance, during two, or three, or

$\frac{x^2}{3a}$; which when $GP = GB$, becomes $OG + \frac{BG^2}{3OG}$; so that the centre of oscillation is at C , viz. lower than G by the quantity $\frac{BG^2}{3OG}$.

Example 3. Let the position be exactly as in the preceding example, excepting only that the axis of suspension or of vibration, which was then perpendicular, be now parallel, to the line AB , as in fig. 7, Plate V. and in this case, the centre of oscillation will coincide with the centre of gravity G ; for here, all the parts of the given line, as A , G , P , B , &c. are equidistant from the axis of suspension; so that the weight x multiplied by the square of its distance from the axis of vibration DOC , becomes a^2x ; the fluent of which is a^2x , and this fluent divided by ax , quotes a ; that is OG for the distance of the centre of oscillation.

Example 4. Let the pendulum consist of an isosceles triangle ABC , fig. 8, Plate V. suspended at A , and the axis of vibration parallel to BC . Put the altitude $AD = a$; base $BC = b$; and $AF = x$. Through F draw GH parallel to the base. Then $a : b :: x : \frac{bx}{a} = GH$; and $\frac{bx\dot{x}}{a}$ is its fluxion, which multiplied by the square of AF , viz. by x^2 , becomes $\frac{bx^3\dot{x}}{a}$. The fluent of this expression is $\frac{bx^4}{4a}$.

The

or four minutes, and then taking the half, or third part, or fourth part of the number; for that part will

The triangle $ABC = \frac{ab}{2}$: and the distance of its centre of gravity from A is $= \frac{2a}{3}$; hence the product of the triangle by this distance is $\left(\frac{ab}{2} \times \frac{2a}{3}\right) = \frac{a^2b}{3}$. Therefore divide the above fluent, $\frac{bx^4}{4a}$, by $\frac{a^2b}{3}$, and the quotient is $\frac{3x^4}{4a^3}$; which when x is equal to the altitude $AD = a$, becomes $\frac{3a}{4}$; so that the distance of the centre of oscillation from A is equal to $\frac{3}{4}$ of the altitude of the triangle.

Example 5. Let the pendulum consist of a spherical body suspended at O, fig. 9, Plate V. by means of a line OD, which line weighs so little with respect to the body, that its weight may be considered as $= 0$. Imagine DERD to be a section of the sphere through its axis, and perpendicular to the axis of oscillation KL. GE the radius perpendicular to DR. G the centre of gravity, and V the centre of oscillation.

Let SFPS be any concentric circle; and put the ordinate $PM = y$; $GP = x$; $c =$ the circumference of a circle whose radius is one, and draw NM parallel to GR. Suppose a cylindric surface to stand on the circumference SFPS, and to be terminated by the surface of the sphere; then the circumference $SFPS = cx$, and the just mentioned cylindric surface will be $= 2cyx$.

The distance of the particles in each section of this cylindric surface, from the centre of gravity of the section,

will be the number of vibrations answering to one minute.

With

or of a line passing through G parallel to KL, is $GP=x$; therefore the fluxion of the weight or surface is $2cyx\dot{x}$, which multiplied by the square of the distance GP, viz. by x^2 , gives the fluxion $2cyx^3\dot{x}$; whose fluent is $\frac{cyx^4}{2}$.

In order to expunge from this expression one of the variable letters, it must be considered that in a circular arch the sine is to the cosine as the fluxion of the latter is to the fluxion of the former; for in fig. 10, Plate V. where $BE=y$ is the sine; $AE=x$ is the cosine; if you draw CF indefinitely near and parallel to BE , and BD parallel to AG , BD becomes \dot{x} , or the fluxion of the cosine, and CD becomes \dot{y} , or the fluxion of the sine; and since the right-angled triangles ABE , BCD , are equiangular (the angles CBD , EBA , being equal, because each of them is the complement of ABD to a right angle) and similar, we have AE to EB , as CD to DB , viz. $x:y::\dot{y}:\dot{x}=\frac{y\dot{y}}{x}$. Also if the radius be called a ; since the square of AB , or a^2 , is equal to x^2+y^2 ; $x=\sqrt{a^2-y^2}$. Now by substitution the fluent $\frac{cyx^4}{2}$ becomes $\frac{2}{3}ca^2y^3-\frac{2}{5}cy^5$ (for the fluxion of the former; viz. $2cyx^3\dot{x}=2cyx^3\times\frac{y\dot{y}}{x}=2cyx^3\times\frac{y\dot{y}}{\sqrt{a^2-y^2}}=2cyx\sqrt{a^2-y^2}^{\frac{3}{2}}\times\frac{y\dot{y}}{\sqrt{a^2-y^2}}=2cy^2\dot{y}\sqrt{a^2-y^2}=2ca^2y^2\dot{y}-2cy^4\dot{y}$; and the fluent of this last expression is $\frac{2}{3}ca^2y^3-\frac{2}{5}cy^5$). And when $y=a$ = radius, this fluent becomes $\frac{2}{3}ca^5-\frac{2}{5}ca^5=\frac{4}{15}ca^5$.

The

With that number of vibrations, performed in one minute, the distance of the centre of oscillation from

The solidity of a sphere, whose radius is a , is expressed by $\frac{2}{3}ca^3$, which multiplied by the distance $GO=d$, in fig. 9. becomes $\frac{2}{3}ca^3d$.

Lastly, divide the above fluent by the last product; viz. divide $\frac{4}{15}ca^5$ by $\frac{2}{3}ca^3d$, and the quotient $\frac{2a^2}{5d}$ is the distance of the centre of oscillation V from the centre of gravity G , and of course $OV = d + \frac{2a^2}{5d}$.

Should the point of suspension be situated close to the surface, as at D ; then the distance between the centres of suspension and of gravity would become equal to radius, viz. $d=a$; and in that case the distance between the centres of oscillation and of gravity will be $\frac{2a}{5}$; and the distance between the centres of oscillation and suspension will be $\frac{7a}{5}$; that is $\frac{7}{10}$ of the diameter of the sphere.

It plainly appears from the foregoing explanations and examples, that when the string of a pendulum is shortened, every thing else remaining unaltered, the centre of oscillation changes its place; unless indeed the weight of the pendulum or bob be supposed to be condensed in one point, which case can have place only in the imagination.

Consequently what has been demonstrated respecting the cycloidal pendulum must be considered as a matter merely of useful speculation, since from it we derive the time in which a circular pendulum performs its vibrations. But in

from the point of suspension is determined by means of the following easy calculation.

Divide sixty seconds by the number of vibrations, which the pendulum in question has performed in one minute, and the quotient is the time of one vibration. Square this time, (viz. multiply it by itself) and multiply its square by the length of the pendulum that vibrates seconds, viz. by 39,1196 inches, and the last product shews the distance in inches of the centre of oscillation or percussion from the point of suspension in the pendulum in question.

Example 1. Let a cylindrical stick AB, fig. 3, Plate V. of about a yard in length, be suspended at A, and be caused to vibrate. Having observed that it performs 76 vibrations in a minute, it is required thereby to find the distance of its centre of oscillation from the point of suspension A.

Divide 60 seconds by 76 vibrations, and the quotient, 0",79 nearly (viz. 79 hundredths of a second) is the time in which the pendulum in question performs one vibration. Then since the lengths of pendulums are as the squares of the times of vibration; therefore say as the square of one
second.

practice a cycloidal pendulum would not perform all its vibrations in equal times; because by the application of the string to the cycloidal cheeks, the free part of the string would be shortened, and the centre of oscillation would change its place continually.

second, which is one, is to the square of 0,79 hundredth parts of a second, viz. 0,6241; so is the length of the pendulum which vibrates seconds, viz. 39,1196 to the length sought; that is, $1 : 0,6241 :: 39,1196$; where since the first number is unity, you need, according to the preceding rule, only multiply 39,1196 by 0,6241; and the product 24,4 is the distance sought; so that the centre of oscillation C in the stick AB is 24 inches and 4 tenths distant from its extremity A; viz, about two thirds of its length.

Example 2. An irregular body suspended by one end has been found to perform 20 vibrations in a minute. Required the distance of its centre of oscillation from the point of suspension?

Here the time of one vibration is $(\frac{60}{20})$ 3 seconds; the square of which is 9; and 39,1196, multiplied by 9, gives 352,0764 inches, or nearly 29 feet, for the distance sought.

CHAPTER XII.

OF THE MECHANICAL POWERS.

THE preceding chapters contain the doctrine of motion in a manner rather extensive for an elementary work. The abstract mode in which this subject has been delivered, may possibly have deterred the novice from the study of natural philosophy. Perhaps he expected that after every theoretical chapter his attention should be relieved by some experimental application of the doctrine. But if such had been the plan, either the work would have been protracted to an immoderate length, or many useful branches of the theory would have been suppressed.

The importance of the doctrine of motion, and its being the foundation of almost all the phenomena of nature, were the motives which placed it before every other branch of natural philosophy, and the reader may perhaps be pleased to hear, that whoever understands the leading principles of the foregoing theory, will meet with very little difficulty in the perusal of the following parts of philosophy,

losophy. He will also find that the doctrine of motion, which he may formerly have looked upon as a difficult and almost a useless subject of speculation, is of general and extensive application. Every tool, every engine of art, every æconomical machine, all the instruments of husbandry, and of navigation, the celestial bodies, &c. are constructed, and act conformably to the laws of motion.

The knowledge of this doctrine answers two extensive objects. It serves to explain natural appearances, and it furnishes the human being with useful machines, which enable him to accomplish such effects, as without that assistance would be utterly out of his power.—The application to natural phenomena will be instanced in almost every chapter of this work.—The second object will be considered immediately.

Mechanics, in its full and extensive meaning, is the science which treats of *quantity*, of *extension*, and of *motion*. Therefore it considers the state of bodies either at rest or in motion. That branch of it which considers the state of bodies at rest, as their equilibrium when connected with one another, their pressure, weight, &c. is called *Statics*. That which treats of motion, is called *Dynamics*. Both those expressions are, however, used in treating of solid bodies; for the mechanics of fluids has two denominations analogous to the above. It is called *Hydrostatics*,

Hydrostatics, when it treats of the equilibrium or quiescent state, and *Hydrodynamics* or *Hydraulics*, when it treats of the motion, of fluids.

What belongs exclusively to fluids will be noticed in the second part of these elements. The equilibrium of solids has been sufficiently examined in the preceding pages, and will be taken farther notice of in the following; since in treating of motion, of actions, of forces, &c. it will naturally appear that when those forces are equal and opposite to each other, an equilibrium takes place.

The active application of the doctrine of motion consists in the construction of machines for the purposes of overcoming resistances, or of moving bodies. Thus if a man wish to remove a stone of a ton weight from a certain place, for which purpose he finds his strength inadequate, he makes use of a long pole, which being applied in a certain manner, actually enables him to move the stone. Thus also another person may wish to convey some heavy article to the top of his house, he makes use of a set of pulleys with a rope, &c. and by that means easily accomplishes his object.

Infinite is the number, and the variety of machines; but they all consist of certain parts or simple mechanisms, variously combined and connected with each other. Of those simple machines we can reckon no more than six or at most seven; viz. the *Lever*, the *Wheel and Axle*, the moveable *Pul-*

ley

ley or *System of Pulleys*, the *Inclined Plane*, the *Wedge*, and the *Screw* *.

The action or the effect of every one of those mechanical powers, depends upon one and the same principle; which has been fully explained in chapter IV, V, and VI; but we shall for the sake of perspicuity briefly repeat it in the following three or four paragraphs, wherein the attentive reader will find the principles or analysis of all sorts of machines.

The force or momentum of a body in motion, is to be derived not merely from its quantity of matter, or only from its velocity, but from both conjointly; for the heavier any body is, the greater power is required to stop it or to move it; and on the other hand the swifter it moves, the greater is its force, or the stronger opposition must be made to stop it. Therefore, the force or momentum, is the product of the weight or quantity of matter by the velocity. Thus if a body weighing 10 pounds
move

* The writers on mechanics do not agree with respect to the number of the mechanical powers. Some exclude the inclined plane from the number; whilst others reckon it one of the principal, and consider the wedge and the screw as only species of it. The balance has been likewise reckoned a peculiar mechanical power. But it has been rejected by others, either on account of its being nothing more than a lever, or because by the use of a balance no additional power is obtained, which advantage ought in truth to be the characteristic property of a mechanical power.

move at the rate of 12 feet per second, and another body weighing 5 pounds move at the rate of 24 feet per second, their momentums will be equal; that is, they will strike an obstacle with equal force, or an equal power must be exerted to stop them; for the product of 10 by 12, viz. 120, is equal to the product of 5 by 24.

The forces of bodies acting on each other by the interposition of machines is derived from the same principle. Thus the two bodies A and B, fig. 11, Plate V. are connected with each other by the interposition of an inflexible rod AB (the simplest of all machines) which rests upon the prop or fixed point F. If the rod move out of its horizontal situation into the oblique position CFE, the body A will be forced to describe the arch AE, whilst the body B describes the arch BC; and those arches, being described in the same time, will represent the velocities of those bodies respectively; therefore, the momentum of A is to the momentum of B, as the weight of A multiplied by the arch AE, is to the weight of B multiplied by the arch BC.

The velocities of A and B are likewise represented by their distances from F; for the arches AE, BC, are as their radii FA, FB. Those velocities are also represented by the perpendiculars EG, CD; for since the triangles EFG, CDF, are equiangular and similar, (the angles at G and D being right, and those at F being equal) we have EF to FC, as

GE

GE to CD. Therefore the respective momentums of A and B may be represented either by $A \times AE$, and $B \times BC$; or by $A \times AF$, and $B \times BF$; or lastly by $A \times EG$, and $B \times CD$.

Note. The last expression is used when the motion of bodies that are so circumstanced, results from the action of gravity; viz. when one body recedes from, whilst the other approaches, the centre of the earth; because gravity acts in that direction.

This is the principle of all sorts of mechanisms; so that in every machine the following particulars must be indispensably found. 1st. One or more bodies must be moved one way, whilst one or more bodies move the contrary way. One of those bodies or sets of bodies is called the *weight*, and the other is called the *power*, or they may be called *opposite powers*. 2dly. If the product of the weight of one of those powers, multiplied by the space it moves through in a certain time, be equal to the product of the weight of the opposite power multiplied by the space it moves through in the same time; then the opposite momentums being equal, the machine will remain motionless. But if one of those products or momentums exceeds the other, then the former is said to preponderate, and the machine will move in the direction of the preponderating power; whilst the opposite power will be forced to move the contrary way. And the preponderance is represented by the excess of one momentum over
the

the other; for instance, if one of the above-mentioned products or momentums be 24, and the other 12, then the former is said to be double the latter; or that the former is to the latter as two to one.

By a strict adherence to those particulars, the attentive reader will be enabled to estimate the power and effect of every machine, excepting, however, the obstruction which arises from the imperfection of materials and of workmanship; as will fully appear from the following paragraphs.

In the explanation of the properties of the mechanical powers, we suppose the rods, poles, planes, ropes, &c. to be destitute of weight, roughness, adhesive property, and any imperfection; for when the properties of those powers have been established, we shall then point out the allowances proper to be made on the score of friction, irregularity of figure, &c.

THE LEVER.

A lever is a bar of wood, or metal, or other solid substance, one part of which is supported by or rests against a steady prop, called the *fulcrum*, about which, as the centre of motion, the lever is moveable.

The use of this machine is to overcome a given obstacle, by means of a given power.—Thus if the stone A, fig. 12, Plate V. weighing 1000 pounds, be required to be lifted up (so as to pass a rope under

der it, or for some other purpose), by means of the ordinary strength of a man, which may be reckoned equal to 100 pounds weight; a pole or lever CE is placed with one end under the stone at E; it is rested upon a stone or other steady body at B, and the man presses the lever down at C. In this case the man's strength is equal to the tenth part of the stone's weight, therefore its velocity must be ten times greater than that of the stone; that is, the part BC of the lever must be ten times as long as the part BE, in order that the power and the weight may balance each other; and if CB is a little longer than ten times BE, then the stone will be raised. Indeed in this case the part CB needs not be so long; for as the stone is not to be entirely lifted from the ground, a lesser momentum is required on the part of the power at C.

In general, to find the proper length of the lever, we need only multiply the weight by that part of the lever which is between it and the fulcrum; and divide the product by the power; for the quotient will be the length BC, which is necessary to form an equilibrium, and of course a little more than that length will be sufficient to overcome the obstacle.

If when the length of the lever is given, you wish to find what power will be necessary to overcome a known obstacle or weight; multiply the weight by that part of the lever which is between

it and the fulcrum, then divide the product by the other part of the lever, and the quotient is the answer.

The possible different situations of the weight, the fulcrum, and the power, are not more than three; hence arise three kinds of levers; to all of which, however, the preceding calculations are equally applicable. Those species are, 1. when the fulcrum is placed between the weight and the power, as the one already described. 2. When the fulcrum is at one end, the power at the other end, and the weight between them, as in fig. 13, Plate V. And 3. When the fulcrum is at one end, the weight at the other end, and the power between them, as in fig. 14, Plate V.

Some writers add a fourth species, viz. the bent lever; but as this differs only in shape from the others, it does not constitute a proper difference of kind.

Hitherto we have supposed that the weight and the power act in directions perpendicular to the arms of the lever; but when this is not the case, the distances of the power and of the weight from the centre of motion must not be reckoned by the distances of the points of suspension from that centre, but by the lengths of the perpendiculars let fall from the centre of motion on the lines of the direction of the forces. For instance, in fig. 15, Plate V. the power at P, acts by means of the string PB, on the end B of the lever, in a direction BP,

PB, oblique to the lever; and in estimating the momentum of the power, you must multiply the force or power applied to the string, not by the length CB, but by the length CD, of the perpendicular, let fall from the centre of motion C, on the line BP, which is the *line of direction* of the power.

Thus also in the bent lever ABC, fig. 16, Plate V. whose centre of motion is at B; the momentums of D and E are the weight of D multiplied by BG, and the weight of E multiplied by BF. The reason of the last remark is easily derived from the composition and resolution of forces (see chap. VIII.) Therefore we may in general say, that *in any sort of lever, and in whatever directions the power and the weight act on it, if their quantities be inversely as the perpendiculars let fall from the centre of motion on their respective directions, they will be in equilibrio; that is, balance each other.*

It will be hardly necessary to remark, that when the lever is loaded with several weights at different distances from the centre of motion, the momentum on each side of the centre of motion is equal to the sum of the products of all the weights on that side multiplied each by its distance from the centre of motion. Thus in fig. 17, Plate V. the momentum of the side AD is equal to the sum of the products of E multiplied by DA, F multiplied by GA, and H multiplied by OA; and the momentum of the side AB is equal to the sum of C multiplied by BA, and K multiplied by LA.

The use of the lever is so general and so extensive, that levers of all sorts and varieties are to be found in almost every mechanism;—in the works of nature as well as those of human ingenuity.

The bones of a human arm, AC, fig. 18, Plate V. and indeed the greatest number of the moveable bones of animals, are levers of the third kind. In fig. 18, D is the centre of motion; the power (viz. the insertion of the muscle BC, the contraction of which moves the arm) is at C, and the effect is produced, or the weight is lifted, at A.

In this natural lever the power is not advantageously situated; for as it lies very near the centre of motion, it must be much greater than the weight which is to be lifted at A. But the loss of power is abundantly compensated by other advantages, the principal of which is the compactness of the limb.

The *iron crow*, fig. 19, Plate V. which is commonly used by carpenters, blacksmiths, stone-masons, &c. is a bent lever, flattened at A. It is bent a little in order that the weight may be less apt to slip off; and it is flattened for the purpose of its being more easily admitted into narrow crevices.

The *common balance*, fig. 20, Plate V. or pair of scales, is a lever, whose fulcrum or centre of motion is in the middle, and the weights are suspended at the two extremities; but as those extremities are equidistant from the fulcrum, the velocities of the weights are equal; and of course when neither end of the beam preponderates, the opposite weights

must

must be equal ; but when one of the weights exceeds the other, then that arm to which the former is suspended, will descend, &c. And this is all the use that can be made of the balance, viz. to find when two weights are equal or unequal*.

* Common balances are subject to many imperfections, the principal of which are as follows :

1st. A balance is frequently in equilibrio, when the opposite weights in its scales are not equal. This arises from the points of suspension being not equidistant from the centre of motion ; in which case the empty scales may be made to balance each other ; yet when equal weights are put in them, those weights will not balance each other ; for as they are suspended at unequal distances from the centre of motion, their momentums are actually unequal.

2dly. The beam is frequently made too slight ; in which case it is apt to be bent more or less by the weights that are put into the scales ; and of course the apparent equilibrium cannot be depended upon.

3dly. Balances seldom are sufficiently sensible. This defect arises from various causes, as from the great weight of the beam, from roughness and friction at the point of suspension, from the centre of gravity of the beam being considerably below the centre of motion, &c.

Balances have been made in this country and elsewhere, of a wonderful degree of sensibility ; viz. capable of having their equilibrium disturbed by so small a quantity as $\frac{1}{1000000}$ part of the weight in each scale. See the *Journal de Physique*, vol. 33d. and the Phil. Transf. for the year 1798, p. 148. And I have heard of scales even of a greater degree of sensibility.

The *steelyard*, fig. 21, Plate V. (which many writers called by the latin name *statera romana*) is a lever of the first kind, whose fulcrum or centre of motion is at A; the weight B is suspended always at the same distance CA from the centre of motion; but the power or counterpoise E may be shifted from one point to another all along the arm AD; by which means a great variety of weights may be balanced by the same counterpoise E, whose momentum increases in the proportion of its distance from A. The whole length of the arm AD is marked with numbered divisions, each of which indicates the weight of B, which is balanced by the counterpoise E, when E is placed at that particular division. Those divisions are ascertained by trial; for the two arms of the steelyard being unequal in weight, their momentums, when loaden with the weights B and E, cannot be estimated merely by the products of those weights multiplied each by its distance from A.

The steelyard was rendered more perfect by Mr. B. Martin, a philosophical instrument maker of very distinguished ability, who fixed a weight C to the short end of the beam (as is shewn in fig 22, Plate V.) capable of just balancing the opposite arm AD; in which case the momentums of E and B are equal to their weights multiplied by their respective distances from A; consequently the divisions on the arm AD may be easily determined by measurement,

When

When a lever is supported at its two extremities A and C, fig. 23, Plate V. and the weight W is suspended at a point B between A and C, those two points A and C may be alternately considered one as the power and the other as the fulcrum; from which consideration it appears that the proportion of the weight which is supported by one of those props, is to the other in the inverse proportion of the distances AB, BC; hence when a weight is carried by means of a pole between two men, in the manner commonly practised by draymen when they carry a cask of beer, the weight may be made to bear harder upon one of the men than upon the other; by placing it nearer to the one than to the other,

THE WHEEL AND AXLE.

The wheel and axle (by some called *axis in peritrochio*) consists of a cylinder, AB, fig. 1, Plate VI. and a wheel DF fastened to the cylinder, and all moveable round the common axis, which is supported at its two ends B and G.

In this mechanical power, the weight C is raised by a rope which coils about the axle, and the power E is applied to the circumference of the wheel. Here it is plain that the velocity of the weight is to the velocity of the power, as the circumference of the axis is to the circumference of the wheel, or (because circles are as their diameters) as the diameter of the axle is to the diameter of the wheel;

hence the equilibrium in this mechanism takes place when the weight of E is to the weight of C, as the diameter of the axle is to the diameter of the wheel.

Instead of the power E, the wheel may be furnished with little handles or spokes, as represented in the figure, which may be moved by hand. Or long spokes may be fixed through the axis, and the hands of one or more men may be applied to the ends of those spokes, as in fig. 2, for the effect will be the same as if there were a wheel; which is so evident as not to need any farther illustration.

Cranes for raising great weights, capstans, and windlasses, such as are used on board of ships, are engines of this sort.

Fig. 3, Plate VI. represents a very powerful engine, nearly of this sort. ABKI, and CIDH, are two cylinders of unequal diameters, (but the difference of those diameters must not be very great) firmly connected together and moveable by means of the handle F round the common axis EG, whose extremities rest upon two supports. The same rope is fastened with one end at D, and is wound round the small cylinder CD; then it descends and passes round the pulley Z, to the frame of which the weight W is suspended; and lastly, the other end of the rope is fastened at A to the larger cylinder. Now by moving the cylinders round, the rope will unwind itself from the small cylinder, and will coil itself round the large cylinder, as is clearly shown

shewn by fig. 4, which represents the cylinders as seen by an eye placed in the direction of the axis. If the cylinders were of equal diameters, the lower part Z of the rope, or the weight W which is suspended to it, would not be moved; for in that case, as much of the rope as is disengaged from one cylinder at each revolution, would be coiled round the other cylinder; but the cylinders being of unequal diameters, it is evident that at each revolution of the handle F, more of the rope will be coiled round the cylinder ABIK than will be disengaged from the cylinder CDIH; and of course the weight W will be raised.

THE MOVEABLE PULLEY, OR SYSTEM OF
PULLEYS.

The pulley is a thick circular piece of wood, or metal, or other solid matter, moveable round a centre pin or axis, which is fixed in a block or frame, in the manner represented by A. fig. 5, Plate VI. In this fig. the frame is fastened to a steady beam; a rope is passed over the pulley, to one end of which the weight W is suspended, and the power P is applied to the other end of the rope. In this case it is evident, that in order to raise the weight, the power P must move downwards as much as the weight W moves upwards; or in other words, that their velocities are equal; hence no advantage is gained by this mechanism, excepting the convenience of changing the direction of the motion;
so

so that the action of this pulley is exactly analagous to that of the balance. Therefore the third mechanical power is not said to be the pulley in general, but it is said to consist of a *moveable pulley*, or *moveable pulleys*, as shewn in the figures 6, 7, and 8, Plate VI. for in those cases, power is evidently gained.

In fig. 6, the rope is fastened to the hook at F; it passes round the pulley BD, to the block of which the weight W is suspended, and is then held by the power at E. When the power pulls the rope, the block, with the weight, are raised, and the rope is shortened on both sides; for instance, when the pulley, block, &c. are at the dotted situation *m*, the rope has been shortened of the lengths AB, CD; viz. double the height *mD*; and that quantity of rope has been drawn by the power; therefore in order to pull the weight up from the situation W to that of the dotted representation, the power must have moved through twice that space; that is, with double the velocity of the weight; hence the equilibrium in this case takes place when the power is to the weight as one is to two.

Fig. 7, represents the same case, excepting only that in this the direction of the power E is changed by the interposition of a fixed pulley F; so that if W weigh two pounds, the power, or opposite weight E, must weigh one pound to balance it; and then, if a little more weight be added to E, the weight W will be raised.

In

In fig. 8, there is a block or frame containing three pulleys, and having the weight W fastened to its hook; there is also another block fastened to a steady beam, and containing three other pulleys. The same rope passes through them all, and is fastened with one end to the upper block, whilst the power E is applied to its other end. Here it is evident, that in raising the weight, the rope must be shortened at a, b, c, d, e , and f ; viz. six times as much as the weight is raised; and of course the power E must move with six times the velocity of the weight; therefore the equilibrium takes place when E is the sixth part of W ; viz. if W weighs six pounds, E needs not weigh more than one pound, in order to balance the weight W ; but if E weigh a little more than one pound, then the weight W will be raised*. The like reasoning may be extended to any other number of pulleys.

Fig.

* The circumferences of pulleys are generally grooved, or hollowed, in order to receive and retain the rope. The axis, or centre pin, is sometimes fixed to the block, and the pulley moves round it; and at other times the axis is fixed in the pulley, and its two ends move in two holes made in the block.

A great degree of friction is the principal defect to which this mechanical power is liable, and which arises from three causes; viz. from the diameter of the axis bearing a considerable proportion to that of the pulley, from the pulley's rubbing against the sides of the block, and from the ropes not being sufficiently pliable.

The

Fig. 9, of Plate VI. represents another variety of this mechanical power. It consists of one fixt, and one moveable pulley; but in fact each of those pulleys performs the office of three pulleys, for it consists of three grooves of unequal diameters, as is shewn by the lateral representation of one of them at R. The same rope which is fastened with one extremity to one of the blocks, passes successively over the fix grooves, and the power is applied at its other end E.

In order to understand the action of this construction, it must be considered, that in the combination of fig. 8, where the pulleys are all of the same diameter, each pulley must move faster than the preceding pulley, because a greater length of rope must pass over each pulley than over the preceding pulley, as may be easily comprehended by inspecting

The principal contrivances, which have been made for the purpose of diminishing those causes of obstruction, will be mentioned in the next chapter.

In the description of this mechanical power we have considered the ropes as acting always perpendicular to the horizon; but when that is not the case, as for instance, it would be, if in fig. 7, Plate VI. the hook S and the pulley F were placed at a greater distance from each other; then the velocity of the weight is to be estimated not by the length of the rope which is drawn, but by the perpendicular height to which the weight is raised. And the same thing must be understood with respect to the direction of the power.

inspecting fig. the 7th, where it is evident, that if the weight be raised one foot, the ropes must be shortened of a foot each, viz. a foot from B to A, and another foot from D to C; hence whilst the length A B passes over the pulley B D, twice that length must pass over the pulley F; so that the pulley F, if equal in diameter to B D, must make two revolutions, whilst the pulley B D makes one revolution. It is also evident, that if the pulley F were of double the circumference, or, which is the same thing, of double the diameter of B D, then each of the pulleys would make one revolution in the same time. Now returning to the construction of fig. 9, it will be easily comprehended, that as the three grooves of the upper pulley, as also the three of the lower pulley, belong to one solid body, they must revolve in the same time; therefore, their diameters, or their circumferences, must be made in the proportion of the quantity of rope, which must pass over them in the same time. Thus whilst one foot length of the rope passes over the first groove *a*, two feet of rope must pass over the second groove *b*, three feet of rope must pass over the third groove *c*, and so forth. Therefore, the diameter of the second groove *b*, must be twice the diameter of the first groove *a*; the diameter of the third groove *c*, must be three times that of the first *a*; the diameter of the fourth groove *d*, must be four times that of *a*; &c. or, in other words,
the

the diameters of the grooves a, b, c, d, e, f , must be in arithmetic progression; the difference of the terms being equal to the diameter of the first or smallest groove.

It is evident, that in this construction, in order to raise the weight, six ropes must be shortened, and of course the power must move through six times the space that the weight moves through, consequently the equilibrium takes place when the power is equal to the sixth part of the weight W .

The principal advantage which is attributed to this construction, is the reduction of friction; for in this, there are only two axes and four surfaces which rub against the blocks; whereas in the construction of fig. 8, where the pulleys are all separate, there are six axes and 12 surfaces which rub against the blocks. But, in my opinion, this advantage is more than compensated by the imperfections which are peculiar to this construction; for, in the first place, if the grooves are not made exactly in arithmetic progression, or if they become otherwise by the accumulation of dirt, &c. then the rope must partly slide over them, which will occasion a considerable degree of friction; and secondly, even when the grooves are of the proper dimensions, if the rope happens to stretch more in one place than in another, which is generally the case, then the above-mentioned sliding and friction will also take place.

THE INCLINED PLANE.

A plane superficies inclined to the horizon, is another mechanical power; its use being to raise weights from one level to another, by the application of much less force than would be necessary to raise them perpendicularly. Thus in fig. 10, Plate VI. AB represents a plane inclined to the horizontal plane AC; where if the weight D be rolled upwards from A to B, the force necessary for the purpose will be found to be much less than that which would be required to raise it directly and perpendicularly from C to B.

In this case the effect which is produced, consists in the raising of the weight from the level of AC to the level of B; but to effect this, the power must have moved from A to B; (for the power acts in that direction, whilst the weight or gravity of the body acts in the direction of the perpendicular CB;) therefore the velocity of the weight in this engine, being to the velocity of the power, as the perpendicular height BC of the plane is to its length AB, the equilibrium takes place when the weight is to the power, as the length of the plane is to its perpendicular height.

This property may be clearly shewn by the following experiment:—Let AB, fig. 11, Plate VI. be a plane moveable upon the horizontal plane AC; viz. so as to admit of its being placed at any required

quired angle of inclination, which is easily accomplished by means of a hinge at A, and a prop between the two planes. The upper part of the plane must be furnished with a pulley B, over which a string may easily run. Let the cylindrical weight D be made to turn upon slender pins in the frame F, in which the hook *e* is fastened with a string *e* BH, which passing over the pulley B, holds the weight E suspended at its other extremity.—The pulley should be situated so that the rope *e* B, may be parallel to the plane.

This plane may be fixed at any angle of inclination, and it will always be found, that if the weight of the body E be to the weight of the body D, together with that of its frame F, as the perpendicular height CB of the plane is to its length AB, the power E will just support the cylinder D, with its frame F upon the plane, and the least touch of a finger will cause the cylinder D to ascend or descend; the counterpoise or power E moving at the same time the contrary way.

It is evident, that the smaller the angle of inclination is, the less force is required to draw up the weight D; and of course when the angle of inclination vanishes or becomes nothing, the least force will be sufficient to move the body; that is, when the plane AB becomes parallel to the horizon, or upon an horizontal plane, the heaviest body might be moved with the least power, were it not for the friction

friction, which is occasioned by the irregularity of the contiguous surfaces, &c. (1.)

THE

(1.) The above-mentioned explanation of the property of the inclined plane, applies only to one direction of the power; namely when the power acts in a direction parallel to the plane; but the general theory will be found in the following proposition:

When a body or weight W is sustained upon a plane, which is inclined to the horizon; viz. when the power P is just sufficient to balance the weight upon that plane; then the power is to the weight, as the sine of the plane's inclination is to the sine which the direction of the power makes with a line perpendicular to the plane.

Let AB fig. 12, Plate VI. be the plane inclined to the horizon AC, and let a weight at O be supported partly by the plane, and partly by a power which acts in the direction OV. Through O draw LOC perpendicular to AB, and at C, where EC meets the horizontal plane, erect CV perpendicular to the horizon, to meet the direction of the power as at V.

Now the body W, situated at O, is balanced, or kept at rest, by three powers, which (see prop. IV. chap. VIII.) have the same proportion to each other as have the right lines parallel to their respective directions, and terminated by their mutual concurrence; namely, by the power which is as OV; by the gravitating power, which is as VC; and by the reaction of the plane, which is as OC; hence the power is to the weight, viz. $P : W :: OV : VC$; or (since the sides of plane triangles are as the sines of their opposite angles) $P : W :: \sin. OCV, \text{ or } BAC : \sin. BOC$; (for those angles are equal since the right-angled triangles BOC, and BAC

THE WEDGE.

The wedge has been justly considered as a species of inclined plane; for it consists of two inclined

have a common angle at B) : $\sin. VOE$, or VOC ; for those angles being the complement of each other to two right angles, have the same sine.

From this proposition the following corollaries are evidently deduced :

1. Since $P : W :: \sin. BAC : \sin. VOC$; it will be $P : \frac{1}{\sin. VOC} :: W : \frac{1}{\sin. BAC}$; therefore if the weight W , and the inclination of the plane, or $\sin. BAC$ remain the same, the power must increase or decrease inversely as the sine of VOC ; hence when the direction of the power is perpendicular to EC , or parallel to the plane AB , then the sine of VOC , being the sine of a right angle, is the greatest sine possible, and, of course in that case the power P , which is required to sustain the weight W , is the least possible; or, which amounts to the same thing, then the greatest weight may be sustained by a given power. Also when the direction of the power coincides with OC , namely when the power acts in a direction perpendicular to the plane, then the angle VOC vanishes, and the power must be infinitely great.

2. If the direction of the power be parallel to, or coincide with the plane, then the equilibrium takes place when the power is to the weight :: $OB : BC ::$ (Eucl. p. 8. B. VI.) $BC : BA$; viz. as the elevation of the plane is to its length; or as the sine of its inclination is to radius.

3. If the direction of the power be OR ; that is, parallel to the horizon, then the equilibrium takes place when the power

inclined planes joined base to base, as shewn in fig. 1, Plate VII. where AB or GC is the thickness of the wedge at its back, upon which the force or power is applied (be it the stroke of a mallet, or any other pressure); the middle line FD is the axis or height of the wedge; DG and DC are the lengths of its flant sides; and OD is its edge, which is to be forced into the wood or other solid; since the use of this instrument is for cleaving of wood, stone, and other solid substances; or, in general, for separating any two contiguous surfaces.

power is to the weight :: OR : CR :: (since the triangles ORC and BAC are similar) BC : CA; viz. as the elevation of the plane is to its base.

4. The power must sustain the whole weight, when its direction is perpendicular to the horizon.

5. The power is to the pressure on the plane :: OV : OC :: sin. OCV : sin. OVC :: sin. BAC : sin. OVC.

6. The preceding analogy, by alternation, becomes P : sin. BAC :: press. : sin. OVC, from which it appears that when the power and the inclination of the plane, or angle BAC, remain invariable, the pressure on the plane must increase or decrease according as the sine of OVC increases or decreases; therefore when the direction of the power is parallel to the base, and OVC becomes a right angle, whose sine is the greatest, then the pressure on the plane will likewise be greatest.

7. When the direction of the power is parallel to the plane, P : pressure :: OB : OC :: BC : AC.

8. When the direction of the power is parallel to the base AC, then P : pressure :: OR : OC :: BC : BA.

Hence its application is very extensive, and in fact, scissars, knives, nails, chisels, hatchets, &c. are nothing but wedges under different shapes.

Strictly speaking, in the geometrical language, a wedge may be called a *triangular prism*; for it may be conceived to be generated by the motion of a plane triangle in a direction parallel to itself, as that of the triangle GCD, from GCD to ABO. And it is called an *isosceles* or *scalene* wedge, according as the generating triangle, or face, GCD, is *isosceles* or *scalene*.

The action of the wedge, is evidently derived from that of the inclined plane; yet a variety of circumstances has rendered the investigation of the power of the wedge more perplexing than that of any other mechanical power*.

The most rational theory shews, 1. *That when the pressures on the sides of the isosceles wedge are equal and act in directions perpendicular to those sides, the equilibrium takes place, when the force on the back of the wedge is to the sum of the pressures on the sides, as GF, viz. half the thickness of the back, is to either of its slant sides, GD, or CD.* 2. *That when the pressures*

* The proportion between the power, which is applied to the back of the wedge, and the effect which is produced on the sides, has been stated differently by different authors. Those who wish to examine the reasons of those different opinions, may consult *Rowning's Comp. Syst. of Phil.* P. I. chap. 10; and *Ludlam's Mathem. Essays*.

forces are equal, but act in directions equally inclined to the sides of the isosceles wedge, the equilibrium takes place when the force on the back is to the sum of the resistances upon the sides, as the product of the sine of half the vertical angle GDC of the wedge, multiplied by the sine of the angle which the directions of the resistances make with the sides, to the square of radius. And 3. that when in a scalene wedge three forces acting perpendicularly upon its three sides, keep each other in equilibrio, those three forces are respectively proportional to the sides.

The three parts of this proposition will be found demonstrated in the note (2).

From

(2) In order to demonstrate the first part of the above-mentioned proposition, let AKD, fig. 2, Plate VII. represent the face of an isosceles wedge. B and E are two obstacles, which press upon its two sides in directions perpendicular to those sides. Suppose the wedge to be impelled downwards as far as the dotted representation GLF, in consequence of which the obstacles B and E must be driven to the places O and M. Through O and M draw OI and MQ parallel to the middle line or axis CD of the wedge; which lines will meet those sides in two points I, Q. Join I, Q, as also O, M, with the lines IQ, OM. Then it is evident from the parallelism of the lines, that OM is equal to IQ; hence the part IQ of the wedge must have advanced as far as OM; therefore YN, or IO, or QM, represents the velocity of the wedge (that is of the power); whilst BO and EM represent the velocities of the obstacles,

From this it follows that by the addition of a little more force on the back of the wedge, than that which is sufficient to form the equilibrium, the resistances will be overcome, &c.

It

Now the triangles IOB, and ACD are equiangular; (the angles at C and B being right, and the angle BIO equal to CDA; Eucl. p. 29. and 32. B. I.) and of course similar (Eucl. p. 4. B. VI.) therefore considering half the wedge and one obstacle, $OB : OI :: AC : AD$; that is, the velocity of the obstacle B is to the velocity of the power, as half the thickness of the wedge is to its slant side. Likewise for the same reasons we say that the velocity of the pressing obstacle E is to that of the power, as half the thickness of the wedge is to its slant side. Therefore, by adding those proportional quantities, we say that the velocity of the obstacle B plus the velocity of the other obstacle E, is to the velocity of twice the half wedge, (viz. of the whole wedge) as the whole length AK of the back, is to the sum of the sides AD, DK; or as half the length of the back is to one side.

But when opposite powers, which act upon each other, are inversely as their velocities, they form an equilibrium; therefore when the power on the back of the wedge is to the sum of the resistances on the sides, as half the length of the back is to one slant side, the wedge remains motionless, which is the first part of the proposition.

In order to prove the second part of the proposition; let ABC, fig. 3, Plate VII. be the face of an isosceles wedge, HC its height or middle line, E and e two obstacles which press upon, or are to be removed, in the directions EF, ef equally inclined to the sides of the wedge. Let the force, represented

It also appears that the smaller the angle GDC is, the less force will be required to drive the wedge into any solid substance.

We

represented by the line EF, be resolved into two other forces; viz. FD parallel, and DE perpendicular, to AC; then the former of those forces, being parallel to the side of the wedge, cannot have any power upon it; therefore the original force EF will have just the same effect upon the wedge as the lesser perpendicular force DE; the former being to the latter as radius to the sine of the inclination of the force EF to the side AC. But, by the first part of this proposition, this perpendicular force DE is to the power on the back of the wedge which balances it, as AC is to AH, or as radius to the sine of the angle ACH, (viz. half the angle at the vertex of the wedge) therefore, by compounding those ratios, $EF \times ED : \text{power on the back} \times ED :: \text{force EF} : \text{power on the back} :: \text{square of radius} : \text{sine of half the vertical angle} \times \text{sine of the inclination of the resistance}.$

The oblique force *ef* on the other side of the wedge, being equal to EF, will require another power equal to the former on the back of the wedge, to balance it; therefore the sum of the resistances on the sides of the wedge is to the whole power on the back, as the square of radius is to the product of the sine of half the vertical angle multiplied by the sine of the inclination of the resistances to the sides of the wedge.

In order to prove the last part of the proposition, let GD, GF, and GE, fig. 4, Plate VII. represent the directions of the three forces perpendicular to the sides of the scalene wedge ABC. Produce EG straight towards O, and through F draw FO parallel to DG. Then since those three forces balance each other, they must be (by prop. IV.

We shall lastly observe, that when the wood splits below the edge of the wedge, as is shewn by fig. 5, Plate VII. which is generally the case; then the side of the wedge must be considered as equal to either side of the cleft; for in fact if we suppose that the wedge is lengthened downwards to the very apex of the cleft; the effect will be the same.

THE SCREW.

The screw is the last mechanical power that remains to be described. This is likewise considered as a species of inclined plane; it being in fact nothing

of chap. VIII.) respectively proportional to the three sides of the triangle GOF; but this triangle GOF is equiangular, and therefore (Eucl. p. 4. B. VI.) similar, to the triangle ABC; therefore the three forces are likewise respectively proportional to the three sides of the triangle or wedge ABC.

The triangles GOF and ABC are equiangular; for the four angles of the quadrilateral figure AEGF are equal to four right angles (Eucl. p. 32. B. I.) and since the angles at E and F are right, the two angles FAE and FGE must be equal to two right angles; that is, equal to FGE plus FGO. Therefore taking away the common angle FGE, there remains the angle FAE equal to FGO. Also in the like manner is proved that the angle DBE is equal to the angle OGD, and likewise equal (Eucl. p. 29. B. I.) to FOG. And since the two angles at O and G of the triangle FGO are respectively equal to the two angles at A and B of the triangle ABC; the third angle of the former must be equal to the third angle of the latter.

thing more than an inclined plane coiled round a cylinder ; and the nut or perforated body which moves up or down a screw, moves up or down an inclined plane in a circular, instead of a rectilinear, direction.

Either the screw *A* may be moved forwards and backwards in a fixed nut as in fig. 6, Plate VII. or the screw *A* remains fixt, and the nut *BC*, or perforated piece, is made to move upon the screw as in fig. 7.—By way of distinction *A* is called the *male screw*, and the nut *B* with its perforation shaped like an hollow screw, is called the *female screw*.—The spiral projections *e, f, g*, &c. are called the *threads* of the screw.

The power which moves this most useful, and most powerful engine, is applied either to one end of the screw, which is generally furnished with a sort of head or projection, or to the end of a lever which is fixed either in the head of the screw as in fig. 6. or in the nut *BC*, as in fig. 7. And then indeed it may with more propriety be called an engine compounded of a screw and a lever.

In all cases the equilibrium takes place between the effect which is produced at the end of the screw or at the nut, and the power, when the former is to the latter as the circumference described by the power in one revolution, is to the distance between two contiguous threads of the screw. Thus supposing that the distance between the threads be half an inch and the length of the lever *CD* be 12 inches ;

inches; the circle described by the end D of the lever will be about 76 inches, or 152 times the distance between two contiguous threads; therefore if the power at the end D of the lever be equivalent to one pound, it will balance a pressure of 152 pounds acting against the end of the screw in fig. 6; or it will support a weight of 152 pounds on the board B, fig. 7, &c.

The reason of this is so evidently dependent on the properties of the inclined plane, that nothing more needs here be said about it.

The least reflection on the preceding explanation of the nature and properties of the mechanical powers will sufficiently prove that, strictly speaking, the real and original mechanical powers are not more than two in number; namely, the lever and the inclined plane; so that all the others are only species of those two; the balance, the wheel and axle, and the pulley, being species of lever; and the wedge with the screw being species of inclined plane. It is however immaterial whether those powers be reckoned all primitive and distinct from each other, or not; for the theory remains always true and the same. The only advantage which might be derived from the idea of the original mechanical powers being only two, is that their properties might, in that case, be explained in a much more concise manner; yet it is to be observed that, after a certain limit, theories became obscure and perplexing,

in proportion as they are rendered more concise and comprehensive.

Before we quit the present chapter it will be proper to make the following remark, the object of which is to prevent the establishment of wrong notions in the mind of the reader, with respect to the powers of the above-mentioned engines.

Beginners in this branch of natural philosophy frequently imagine that by means of the mechanical powers, a real increase of power is obtained; whereas this is not true. For instance, if a man be just able to convey 100 weight from the bottom to the top of his house in one minute's time, no mechanical engine will enable him to convey 300 weight to the same height in the same time; but the engine will enable him to convey the 300 weight in three minutes; which amounts to the same thing as to say that the man could, without the engine, carry the 300 weight by going three times to the top of the house, and carrying 100 weight at a time, provided the load admitted of its being so divided. Therefore the engine increases the effect of a given force by lengthening the time of the operation; or (since uniform velocity is proportional to the time) by increasing the velocity of that force or power.

Thus again, if any active force is able to raise a weight of 10 pounds with a given velocity, it will be found impossible, by the use of any instrument,

to

to make the same force raise a weight of 20 pounds, or in general a weight more than 10 pounds, with the same velocity; but it may, by the aid of the instruments, be made to raise the weight of 20 pounds with half that velocity; or, which is the same thing, it may be made to raise it to half the height in the same time; for it is not the power or force, but the momentum, (viz. the product of the force by the velocity) that may be increased or diminished by the use of those engines.

The power, or acting force, is so far from being increased by any machine, that a certain part of it is always lost in overcoming the resistance of mediums, the friction, or other unavoidable imperfections of machines. And this loss in some compound engines is very considerable.

CHAPTER XIII.

OF COMPOUND ENGINES; OF THE MOVING
POWERS; AND OF FRICTION.

ALL the instruments or machines which connect an active force with a certain effect, however complicated they may be, will, upon examination, be found to consist of the already described mechanical powers. Those component simple mechanisms are frequently varied in shape; their connection is infinitely diversified; but their nature and their properties remain invariably the same.

Various are the powers which have been employed as first movers of machines; but the principal of them are, 1. The natural strength of a man or number of men. 2. The strength of other animals, and principally of horses. 3. The force of running water. 4. The force of the wind. 5. The elastic force of the steam of boiling water. 6. The elastic force of springs. 7. The simple weight of heavy bodies.

A great part of most machines relates to the power itself; viz. it consists of contrivances necessary for the generation, application, preservation, and renovation, of the active power or force. The
effect

effect which is to be produced by that active force, is derived from the proper application of the above-mentioned simple mechanical powers ; which divide, or concentrate, or regulate, the original force. As for the effects which are produced by machines, it is impossible to ascertain their number, or even to arrange them under general and comprehensive titles.

The best machine for the production of any particular effect, is that, which (all circumstances of situation, materials, &c. being considered) will produce that effect in the simplest, steadiest, safest, and cheapest manner possible.

It is not my intention to describe the principal machines, that are now in use amongst the enlightened nations of the world ; that being incompatible with the nature and the limits of the present work. Those persons who may be desirous of examining the peculiar constructions of the various engines of arts, manufactures, navigation, economy, &c. will find a great variety of books written expressly on the subject, in almost every language of Europe ; but in none more so than in the French.

In the present chapter, the methods of computing the powers and the effects of machines in general, will be briefly stated ; and the description of a few mechanisms will be inserted merely for the purpose of exemplifying the application of the methods ; whence the reader may be enabled to judge

judge of the power and effect of any other machine that may fall under his examination.

A compound engine either consists of one simple mechanical power repeated two or more times; or, it consists of several simple mechanical powers variously combined, and connected with each other. In any case, the power and the effect must be estimated from the result of the effects of all the component simple mechanisms separately considered, which is done in the following manner :

Find what proportion the power bears to the effect in each simple mechanism; put all those ratios one under the other, and find their sum, which sum will express the proportion between the power and the effect of the whole compound engine*. Thus suppose that a machine is compounded of three simple mechanical powers, viz. a lever, an inclined plane, and a moveable pulley; and suppose that a power applied to one end of the lever will produce a double effect at the other end; for instance, one pound will balance two pounds; then the proportion of the power to the effect, is as one to two. Suppose also that in the inclined plane, the power is

* The sum of two or more ratios is obtained by multiplying the antecedents together and the consequents together; and the two products will form a ratio, which is called the sum of the given ratios. Thus the ratio of 2 to 3, plus the ratio of 2 to 5; plus the ratio of 4 to 7; is equal to the ratio of 16 to 105; viz. of $2 \times 2 \times 4$ to $3 \times 5 \times 7$.

is to the effect as three to seven ; and lastly, that in the pulley, the power is to the effect as one to two. Now those three ratios being written one

under the other thus	$1 : 2$	and
	$3 : 7$	
multiplied. viz. all the antecedents	$1 : 2$	
together, and all the consequents	<hr style="width: 100px; border: 0.5px solid black;"/>	
	$3 : 28$	

together, the two products thence arising will exhibit the proportion which the power bears to the effect in the whole compound engine ; viz. that a power of 3 pounds will balance a weight of 28 pounds.

Otherwise the effect of a compound engine may be computed by considering the velocities of the power and of the effect ; for they are to each other inversely as their velocities, viz. the power is to the effect as the velocity of the latter is to that of the former. Thus in a certain compound engine I find that the power must move through 500 feet, whilst the weight moves through 3 feet ; I therefore conclude that a power of 3 pounds will balance a weight of 500 pounds in that machine, and of course a little more than 3 pounds will raise the 500 pounds weight.

Fig. 8, Plate VII. represents an engine consisting of three levers CD, DG, GH, each of which moves round a pin or axis fixed to a steady post, and are disposed so as to act upon each other. Let CA, DE, GK be each one foot long, and AD, EG, KH

be each three feet long, then the weight I of one pound will produce a pressure of three pounds at G on the end of the next lever, and this force will produce an effect equal to 3 times 3, or 9 pounds, on the end D. Lastly, the end D of the third lever CD being pressed downwards with a force of 9 pounds, will balance or keep suspended at the opposite end C the weight W, of 3 times 9, viz. of 27 pounds. And a little addition of power to the end H will enable the engine to lift up the weight W*.

By increasing either the number of levers, or the difference of length between the two parts of each lever, the effect at the end C may be increased to any degree. But the great defect of this engine is, that the end C with the weight W can be raised a very short way.

* The effect of this compound engine may be calculated according to the preceding rule, by setting down the proportion between the power and the effect in each of the three levers; then multiplying the antecedents together and the consequents together, their products will give the answer. Thus since the two parts of each lever are as one to three, therefore we have - - I : 3

$$I : 3$$

$$I : 3$$

$$I : 27$$

which shews, that in order to form the equilibrium, the power I must be to the weight W as 1 to 27, the same as above.—The same thing may be done with other engines.

Fig. 9, Plate VII. represents a combination of four pulleys, three of which are moveable and one is fixt. But this combination must not be reckoned a single mechanical power, because the same rope does not run over all the pulleys. It is, therefore, a repetition of one and the same mechanical power.

Three ropes are fastened to the beam EF, at G, H, and K. The first rope goes round the pulley A, to the block of which the weight W is fastened; and is then tied to the hook of the block of B. The second rope goes round the pulley B, and is fastened to the hook of the block of C. The third rope goes round the pulley C, as also round the fixt pulley D; and holds the counterpoise or power I.

The pulley D being fixt to the beam, does nothing more than change the direction of the motion; therefore if the power I weigh one pound, it will balance a weight of two pounds affixed to the block of the pulley C. Then the pulley C acting with a power of two pounds, will balance a weight of twice two, or of 4 pounds affixed to the block of the pulley B, and this will balance a weight W of twice 4, or 8 pounds, affixed to the block of the pulley A; so that in order to pull up the weight W of 8 pounds, the power I needs be very little heavier than one pound.

This engine is subject to the same inconvenience as the preceding; viz. the weight W can be raised but a very little way.

Fig.

Fig. 10, Plate VII. represents a combination of five wheels A, B, C, D, E; each of which turns round a centre-pin, which is supposed to be fixed to a steady frame. Those wheels are connected with each other in the following manner: The wheel A has a small wheel or *pinion* o fastened to, and concentric with, itself. This pinion is furnished with teeth, which move between the teeth on the circumference of the next wheel, which is likewise furnished with a pinion which acts in a similar manner on the next wheel, and so on, excepting the last, which has an axle instead of a pinion, and round this axle a rope is applied, to which the weight W is suspended.—The power I is applied to the circumference of the first wheel.

This engine consists of a repetition of the wheel and axle; for the pinion of each wheel is in fact its axle, excepting that instead of acting immediately upon the weight by means of a rope, here it exerts its force against the next wheel by means of its teeth.

Let the circumference of each wheel be equal to five times the circumference of its pinion. Then if a weight I of one pound be suspended to the circumference of the wheel A, the pinion o will act on the circumference of the second wheel with a force equal to five times the power I, viz. equal to 5 pounds, and this force of 5 pounds on the circumference of the second wheel will enable its pinion to act on the circumference of the third wheel

with a force equal to 5 times 5, viz. 25 pounds. After the same manner the force of 25 pounds on the circumference of the third wheel will enable its pinion to act on the circumference of the fourth wheel with a force equal to 5 times 25; viz. 125 pounds, in consequence of which force applied to the circumference of the fourth wheel, the pinion of that wheel will act on the circumference of the last wheel with a force equal to 5 times 125; viz. 625 pounds, which force will balance a weight *W* of 5 times 625, viz. of 3125 pounds. Therefore the power *I* of one pound will balance the weight *W* of 3125 pounds.

Fig. 11, Plate VII. represents an engine compounded of a lever, a screw, and a wheel and axle. The lever *AB* is moved by the application of a hand to the handle *A*. As the lever *AB* turns the axis with the screw *D*, which is all fixed together, the screw *D*, working into the teeth of the wheel *C*, will move this round its axis *E*, in consequence of which the weight *W* will be drawn up or let down according as the lever *AB* is turned one way or the other. Let the power which is communicated by the hand be equivalent to one pound; then if the circumference which is percurred by the handle *A*, be equal to 100 times the distance between two contiguous threads of the screw *D*, this screw will act on the circumference of the wheel *C* with a force equal to 100 pounds; and if the diameter of the wheel *C* be to the diameter of the

the

the axle E as 8 to one, then the power of 100 pounds on the circumference of the wheel C will act with a force equal to 8 times 100; viz. of 800 pounds on the circumference of the axle E, about which the rope of the weight W is wound. Therefore it appears that with this engine a weight or power of one pound will balance a weight W of 800 pounds.

A screw, like D, situated so as only to turn round an axis, but without moving backwards and forwards, and always working on the circumference of a wheel, as C, is usually called *an endless screw*.

Fig. 12, Plate VII. represents an improved crane for raising of goods or heavy weights. This description has been taken from the appendix to Mr. Ferguson's Lectures, which I have preferred to other descriptions of similar engines; first, on account of the improvements it contains, which will naturally shew that a variety of collateral objects must be kept in view by the contrivers of such machines; and secondly, for the purpose of making the reader acquainted with the meaning of the principal terms that are used in mechanics.

A is the great wheel of this engine, and B its axle, on which the rope C winds. This rope goes over a pulley D in the end of the arm of the gib * E, and draws up the weight F, as the

* *Gib*, a projecting transverse beam.

winch* G is turned round. H is the largest trundle†, I the next, and K is the axis of the smallest trundle, which is supposed to be hid from view by the upright supporter L. A trundle M is turned by the great wheel, and on the axis of this trundle is fixed the ratchet-wheel ‡ N, into the teeth of which the catch O falls. P is the lever, from which goes a rope QQ, over a pulley R, to the catch; one end of the rope being fixed to the lever, and the other end to the catch. S is an elastic bar of wood, one end of which is screwed to the floor; and, from the other end goes a rope (out of sight in the figure) to the farther end of the lever, beyond the pin or axis on which it turns in the upright supporter T. The use of this bar is to keep up the lever and prevent its rubbing against the edge of the wheel

* *Winch* or *winder*, an instrument with a crooked handle, the use of which is to turn any thing round.

† A small wheel, which is turned round by the teeth of a large wheel, which derives various denominations from its various shapes. It is called *pinion* when it is oblong, and the teeth are longer than the inside solid part; the teeth are then called the *leaves* of the *pinion*. When the small wheel is shaped like that which is represented at H, it is then called a *trundle*, or sometimes a *lantern*, and even a *drum*.

‡ A *ratchet-wheel* or *ratchet*, is a wheel generally having its teeth bent one way, wherein a solid piece, called a *catch* or *click*, falls, by which means the ratchet-wheel, when the catch bears upon it, can turn one way only, but not the contrary way.

V, and to let the catch keep in the teeth of the ratchet-wheel. But a weight hung to the farther end of the lever would do full as well as the elastic bar and rope.

When the lever is pulled down it lifts the catch out of the ratchet-wheel, by means of the rope QQ, and gives the weight F liberty to descend; but if the lever P be pulled a little farther down than what is sufficient to lift the catch O out of the ratchet-wheel N, it will rub against the edge of the wheel V, and thereby hinder the too quick descent of the weight, and will quite stop the weight, if pulled hard. And if the man who pulls the lever should inadvertently let it go, the elastic bar will suddenly pull it up, and the catch will fall down and stop the machine.

WW are two upright rollers above the axis or upper gudgeon * of the gib E. Their use is to let the rope C bend upon them, as the gib is turned to either side, in order to bring the weight over the place where it is intended to be let down.

N. B. The rollers ought to be so placed, that if the rope C be stretched close by their utmost sides, the half thickness of the rope may be perpendicularly over the centre of the upper gudgeon of the gib. For then, and in no other position of the

* The pins or extremities of an axle, which pins move in holes, &c. are called *gudgeons* in large works, and *pevets* or *pivots* in small works.

rollers, the length of the rope between the pulley in the gib, and the axle of the great wheel, will be always the same, in all positions of the gib, and the gib will remain in any position to which it is turned.

When either of the trundles is not turned by the winch in working the crane, it may be drawn off from the wheel, after the pin near the axis of the trundle is drawn out, and the thick piece of wood is raised a little behind the outward supporter of the axis of the trundle. But this is not material; for, as the trundle has no friction on its axis but what is occasioned by its weight, it will be turned by the wheel without any sensible resistance in working the crane.

This engine is to be situated in a room with the gib E projecting out of it, so that the load may be raised from the street or other lower situation, by turning the winches of the trundles as at G.

This crane has four different powers. The three trundles H, I, K, are furnished with different numbers of staves*; the largest has 24 staves, the next 12, and the smallest 6. The great wheel A has 96 cogs†; therefore the largest trundle makes four revolutions for one revolution of the wheel; the next makes 8, and the smallest makes 16. A

* The sticks or cylindrical bars of trundles, which perform the office of teeth, are called *staves*.

† *Cogs* are the wooden teeth of a large wheel.

winch G is occasionally put upon the axis of either of these trundles, for turning it; that trundle being then used which gives a power best suited to the weight. The length of the winch is such, that in every revolution its handle describes a circle equal to twice the circumference of the axle B of the wheel. So that the length of the winch doubles the power gained by each trundle.

If the winch be put upon the axle of the largest trundle, and turned four times round, the wheel and axle will be turned once round; and the circle described by the power that turns the winch, being in each revolution double the circumference of the axle, when the thickness of the rope is added thereto, the power goes round 8 times as much space as the weight rises through; and therefore (making some allowance for friction) a man will raise 8 times as much weight by the crane as he would by his natural strength without it; the power, in this case, being to the weight as 8 to 1.

If the winch be put upon the axis of the next trundle, the power will be to the weight as 16 to 1, because it moves 16 times as fast as the weight moves.

If the winch be put upon the axis of the smallest trundle, and turned round, the power will be to the weight as 32 to 1.

But, if the weight should be too great even for this power to raise, the power may be doubled by drawing up the weight by one of the parts of a double

double rope, going under a pulley in the moveable block which is hooked to the weight below the arm of the gib. And such is the actual representation of the figure. Then the power will be to the weight as 64 to 1. Whilst the weight is drawing up, the ratch-teeth of the wheel N slip round below the catch or click that falls successively into them, and thus hinders the crane from turning backward, or detains the weight in any part of its ascent, if the man, who works at the winch, should accidentally quit his hold, or choose to rest himself before the weight be quite drawn up.

In order to let down the weight, a man pulls down the end Z of the lever, which lifts the catch out of the ratchet-wheel, and gives the weight liberty to descend. But, if the descent be too quick, he pulls the lever a little farther down, so as to make it rub against the outer edge of the round wheel V, by which means he lets down the weight as slowly as he pleases; and by pulling a little harder, he may stop the weight, if necessary, in any part of its descent. If he accidentally quits hold of the lever, the catch immediately falls, and stops both the weight and the whole machine.

In the construction of machines in general, the questions which occur in the first place, relate to the choice of the power, and to the estimation of its quantity; viz. whether the force of wind or water, or of a man, &c. should be preferred, and what

what is the value or quantity of any one of those powers?

Of the principal active powers which have been enumerated at the beginning of this chapter, we shall briefly mention the common estimation of their forces; but we shall take more particular notice of the force of wind, of water, and of steam, when we come to treat of the properties of air, of the steam of water, &c.

The power which can be applied as the first mover of a machine, in the easiest manner, and whose action is most uniform, is the simple weight, such as is applied to clocks, jacks, and other machines; but this sort of power requires to be renewed after a certain period; that is, it must be wound up, or raised, on which account it is mostly used for slow movements; especially when a very regular action is required.

The force of running water, and that of the wind, where the situation of the place admits of their being used, are very powerful and advantageous movers of machines, such as mills, pumps, sawing engines, &c.—They may be applied to the working of the greatest engines. Running water is preferable to wind, on account of its acting with much more constancy and uniformity.

The steam of boiling water is likewise a most powerful agent; and the recent improvements which have been made by several ingenious mechanics in this country have extended the application of it
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from the smallest to the largest engines. The application of this power requires a very nice construction of the mechanism, and is attended with a considerable consumption of fuel, which particulars are not to be obtained in every situation.

A spring is likewise a useful and commodious moving power ; but a spring, like the weight, requires to be wound, or set up, after a certain time ; viz. when it is quite unbent ; on which account it is more commonly used for slow movements, such as watches, table clocks, &c. But this sort of power differs from the weight in a very remarkable circumstance ; which is, that its action is never uniform. It is strongest when most bent, and it decreases in proportion as it unbends.

In order to remedy this defect, and to render the action of a spring uniform and effectual, a curious contrivance has been long in use, and is as follows : An hollow groove of a spiral form is made round a solid piece of metal, such as is represented at fig. 13, Plate VII. which is furnished with an axis AB, round which it turns in the frame of the machine, and is connected with a wheel *g*, whose teeth act upon the other wheels of the machine. This spiral piece is called the *fusée*, and serves to render the action of the spring equable or uniform. It is connected with the spring by means of a string or chain *F*, one end of which is fastened to the spring which is not seen in the figure, and the other end is fastened to the lowest part *d* of the spiral groove.

When

When the fusee is turned so as to wind the string or chain upon it, the spring is thereby set up, or bent, and when afterwards the machine is left to itself, the force of the spring will, by pulling the chain or string, force the fusee to turn round its axis in a direction contrary to that in which it was wound up. Now when the string bears upon the smallest part *c* of the fusee; viz. nearest to the axis where a greater force is required to produce a certain effect, the spring pulls the chain with its greatest force, because it is then bent most; whereas when the string bears upon the lower and larger part of the fusee, where less force is required to produce the above-mentioned effect, there the spring pulls the string or chain with less force, because then it is bent less. Therefore the decreasing force of the spring is compensated by the increase of power with which the string or chain acts on the axis *AB*; hence the teeth of the wheel *g* act always with the same degree of force upon the next wheel; and thus the motion of the mechanism is rendered uniform. This mechanism will be found almost universally applied to pocket watches and spring clocks.

The natural strength of living animals is the last power that remains to be taken notice of; and here we shall not extend our observations beyond the force of men and horses, concerning which the following particulars are deserving of notice. The different writers on mechanics do not quite agree
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in the estimation of the mean strength of a man; nor is it likely they should, considering how the constitution of men varies according to the difference of climate, of nourishment, and of other circumstances. Upon the whole, a man of ordinary strength is reckoned capable of raising a weight of 600 pounds avoirdupoise ten feet high in one minute, and to be able to work at that rate for 10 hours out of 24; or to do any other work that may be equivalent to it. I am however inclined to think that this estimation is rather above than below the real fact.

By means of a judicious application of the human strength, the effect may in some cases be increased, and on the other hand an improper application of it will diminish the effect. Thus if two men work at a windlass, or axle, by means of handles or levers, they will be able to draw a weight of 70 pounds more easily than one man can a weight of 30 pounds, provided the handles or levers are at right angles to each other.

A man is able to draw horizontally not above 70 or 80 pounds; for in that case he can only employ half the weight of his body.

N. B. Here it is not to be understood that a man cannot draw a cart or carriage that weighs more than 80 pounds; for the weight of the cart is supported by the ground; but we mean, that a man will not be able to draw such a cart as will require more than 80 pounds to move it along.

If a man weigh about 140 pounds, he can exert no greater force in thrusting horizontally at a height even with his shoulders, than what is equal to 27 pounds.

A horse, in general, is reckoned capable of doing as much work as five men.

A horse draws with the greatest advantage when the line of direction is a little elevated above the horizon; and the power acts against its breast.

A horse is reckoned capable of drawing against a resistance of 200 pounds at the rate of $2\frac{1}{2}$ miles an hour, and to continue that exertion for eight hours out of 24.

The strength of the horse, like that of a man, may be rendered more or less efficacious, by means of a proper or improper application of it. In going up a steep ascent, five men can carry a much greater weight than a horse. And in certain ascents, men will be able to carry some weight, where a horse will not be able to carry himself. In mills and other machines, where the circular motion of a horse is employed, the diameter of the circular walk should not be less than 25 or 30 feet; otherwise, the motion is neither very advantageous, nor pleasant to the animal. With respect to the quantity of power, it must be observed, that in the practical application of a moving power to a machine of any sort, it is not enough to employ a power which is barely sufficient to overcome the obstacle, or to produce the effect; but such a
power

power must be applied as will produce the desired effect in the most advantageous manner possible; for instance if a power of a pound, applied to a machine, will produce a certain effect in one hour; whereas if a power of two pounds were applied to the same machine, it would produce the like effect in 20 minutes, it is evident, that the application of the latter power would be more advantageous than of the former; for though the latter power be double the former, yet the time of its performing the operation is less than half the time of the former powers performing the same operation. *So that the most advantageous power for moving a machine is that, which being multiplied by the time of performing a determined effect, produces the least product*.*

With respect to friction, two objects must be observed; viz. the loss of power which is occasioned by it, and the contrivances which have been made, and are in use, for the purpose of diminishing its effects.

A body upon an horizontal plane should be capable of being moved by the application of the least force; but this is not the case; and the principal causes which render a greater or less quantity

* For a farther investigation of the most advantageous application of powers to machines, see Gravefand's Mat. Elem. of Nat. Phil. B. I. chap. 21, and the following scholia; also almost all the writers on mechanics.

of force necessary for it, are, 1st, the roughness of the contiguous surfaces; 2dly, the irregularity of the figure, which arises either from the imperfect workmanship, or from the pressure of one body upon the other; 3dly, an adhesion or attraction which is more or less powerful according to the nature of the bodies in question; and 4thly, the interposition of extraneous bodies; such as moisture, dust, &c.

Innumerable experiments have been made for the purpose of determining the quantity of obstruction, or of friction, which is produced in particular circumstances*. But the results of apparently similar experiments, which have been made by different experimenters, do not agree; nor is it likely they should, since the least difference of smoothness or polish, or of hardness, or in short of any of the various concurring circumstances, produces a different result. Hence no certain and determinate rules can be laid down with respect to the subject of friction.

If a body be laid upon another body, and soon after be moved along the surface of it, a lesser force will be found sufficient for the purpose, than if the body be left some time at rest before it be moved. This arises principally from an actual

* See Mr. Coulomb's Essay in the tenth vol. of the *Mémoires des Savants Etrangers*. And M. de Prony's *Architecture Hydraulique*, § 1089, and following.

change of figure, which is produced in a longer or shorter time according to the nature of the bodies. Thus the maximum of adhesion between wood and wood takes place in a few minutes' time; between metal and metal it takes place almost immediately. A hard and heavy body laid upon a softer one will sometimes continue to increase its adhesion for days and weeks.

When a cubic foot of soft wood of eight pounds weight is to be moved upon a smooth horizontal plane of soft wood, at the rate of three feet per second, the power which is necessary to move it, and which is equivalent to the friction, amounts to between $\frac{1}{4}$ and $\frac{1}{3}$ of the weight of the cube.—When the wood is hard the friction amounts to between $\frac{1}{7}$ and $\frac{1}{8}$ of the weight of the cube.

In general the softer or the rougher the bodies are, the greater is their friction. Yet when two pieces of metal, extremely well polished, are laid one upon the other with an ample surface of contact, they adhere to each other much more forcibly than when they are not so well polished.

Iron or steel moves easiest in brass. Other metals, acting against each other, produce more friction.

The friction, *ceteris paribus*, increases with the weight of the superincumbent body, and almost in the same proportion.

The friction or obstruction which arises from the bending of ropes about machines, is influenced by

by a variety of circumstances, such as their peculiar quality, the temperature of the atmosphere, and the diameter or curvature of the surface to which they are to be adopted. But when other circumstances remain the same, the difficulty of bending a rope increases with the square of its diameter, as also with its tension; and it decreases according as the radius of the curvature of the body to which it is adapted, increases.

Of the simple mechanical powers the lever is the least subject to friction.

In a wheel, the friction upon the axis is, as the weight that lies upon it, as the diameter of the axis, and as the velocity of the motion. But upon the whole, this sort of friction is not very great, provided the machine be well executed.—In common pulleys, especially those of a small size, the friction is very great. It increases in proportion as the diameter of the axis increases, as the velocity increases, and as the diameter of the pulley decreases. With a moveable tackle, or block, of five pulleys, a power of 150 pounds will barely be able to draw up a weight of 500 pounds.

The screw is subject to a great deal of friction; so much so that the power which must be applied to it, in order to produce a given effect, is at least double that which is given by the calculation independent of friction. But the degree of friction in the screw is influenced considerably by the nature of the construction; for much of it is owing

to the tightness of the screw, to the distance between its threads, and to the shape of the threads; the square threads, like those of fig. 14, Plate VII. producing upon the whole less friction than those which are sharp, as in the figures 6 and 7 of the same plate.

The friction which attends the use of the wedge, exceeds, in general, that of any other simple mechanical power. Its quantity depends so much upon the nature of the body upon which the wedge acts, besides other circumstances, that it is impossible to give even an approximate estimate of it.

The friction of mechanical engines does not only diminish the effect, or, which is the same thing, occasion a loss of power; but is attended with the corrosion and wear of the principal parts of the machine, besides producing a considerable degree of heat, and even actual fire; it is therefore of great importance in mechanics, to contrive means capable of diminishing, if not of quite removing, the effects of friction.

In compound engines, the obstruction which arises from friction can be ascertained only by means of actual experiments. An allowance, indeed, may be made for each simple component mechanical power; but the error in estimating the friction of any one single power is multiplied and increased so fast by the other parts, that the estimate generally turns out very erroneous. Besides, much depends on the execution of the work; the quality
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of which cannot be learned but by experience. Novices are generally apt to expect too much or too little from any mechanism.—In general it can only be said, that in compound engines, at least one-third of the power is lost on account of the friction.

The methods of obtaining the important object of diminishing the friction, are of two sorts, viz. either by the interposition of particular unctuous or oily substances between the contiguous moving parts; or by particular mechanical contrivances.

Olive-oil is the best, and perhaps the only substance that can be used in small works, as in watches and clocks, when metal works against metal. But in large works the oil is liable to drain off, unless some method be adopted to confine it. Therefore for large works tallow is mostly used, or grease of any sort; which is useful for metal, as well as for wood. In the last case tar is also frequently used.

In delicate works of wood, viz. when a piece of wood is to slide into or over wood, and when a wooden axis is to turn into wood, the fine powder of what is commonly called *black-lead*, when interposed between the parts, eases the motion considerably, and is at the same time a clean and durable substance

Though olive-oil be the best and the only substance that is used for delicate mechanisms; yet it is

far from being free from objections. Oil, when in contact with brass, is liable to grow rancid, in which state it slowly corrodes the brass. In different temperatures it becomes more or less fluid; but upon the whole it grows continually thicker, and of course less fit to ease the motion of the parts, &c. Trifling as those defects may at first sight appear, they are however of such moment in delicate works, that in the greatly improved state to which watch-work has been brought in this country, the changeable quality of the oil seems at present to be the principal, if not the only, impediment to the perfection of chronometers.

The mechanical contrivances which have been made, and are in use, for the purpose of diminishing the effects of friction, consist either in avoiding the contact of such bodies as produce much friction, or in the interposition of rollers, viz. cylindrical bodies, between the moving parts of machines, or between moving bodies in general. Such cylinders derive, from their various size and application, the different names of *rollers*, *friction-wheels*, and *friction-rollers*.

Thus in mill-work and other large machines, the wooden axes of large wheels terminate in iron gudgeons, which turn in wood, or more frequently in iron or brass, which construction produces less friction than the turning of wood in wood. In the finest sort of watch-work the holes are jewelled, viz. many of the pivots of the wheels, &c. move in
H
holes

holes made in rubies, or topazes, or other hard stone, which when well finished are not liable to wear, nor do they require much oil.

In order to understand the nature of rollers, and the advantage with which their use is attended, it must be considered, that when a body is dragged over the surface of another body, the inequalities of the surfaces of both bodies meet and oppose each other, which is the principal cause of the friction or obstruction; but when one body, such as a cask, a cylinder, or a ball, is rolled upon another body, the surface of the roller is not rubbed against the other body, but is only successively applied to, or laid on, the other; and is then successively lifted up from it. Therefore, in rolling, the principal cause of friction is avoided, besides other advantages; hence a body may be rolled upon another body, when the shape admits of it, with incomparably less exertion than that which is required to drag it over the surface of that other body. In fact we commonly see large pieces of timber, and enormous blocks of stone, moved upon rollers, that are laid between them and the ground, with ease and safety; when it would be almost impossible to move them otherwise.

The form and disposition of friction-wheels is represented by fig. 1. Plate VIII, which exhibits a front view of the axis *d* of a large wheel, which moves between the friction-wheels A, B, C. Here the end *d* of the axis (and the same thing must be

understood of the opposite extremity of the axis) instead of moving in a hole, moves between the circumferences of three wheels, each of which is moveable upon its own axis, and is unconnected with the others. Now if the end *d* of the axis turned in a hole, the surface of the hole would stand still, and the surface of the axis would rub against it; whereas when the axis moves between the circumferences of the wheels A, B, C, its surface does not rub against, but is successively applied to the circumferences of those wheels; so that this sort of motion has the same advantage over the turning of the axis in a hole, that the moving of a heavy body upon rollers has over the simple method of dragging it upon the ground. In this construction the contact of the axis *d*, moves the wheels A, B, C, round their axes, where indeed some friction must unavoidably take place, but that friction is very trifling; for if the circumference of the axis *d* be to that of each wheel as one to 20, the axis must make 20 revolutions whilst the friction-wheels will turn round once only.

A few years ago the same principle was applied in a very ingenious manner, by Mr. John Garnett then of Bristol, to pulleys, and other sorts of circular motion round an axis, for which he obtained a patent. The use of this application has proved very advantageous, especially on board of ships, where it has been found, that with a set of Mr. Garnett's pulleys, three men were able to draw as
much

much weight as five men were barely able to accomplish with a similar set of common pulleys.

One of those pulleys is represented by fig. 2, Plate VIII. where the shaded part BBB is the pulley, A is the axis, and *c, c, c, c, c, c,* are the cylindrical rollers, which are situated between the axis and the inside cavity of the pulley. The ends of the axis A, are fixed in a block, after the usual manner. Every one of the rollers has an axis, the extremities of which turn in holes made in two brass or iron flat rings, one of which is visible in the figure.

After having given a general explanation of the action of rollers, the advantage which Mr. Garnett's pulleys must have over those of the common sort, needs no farther illustration. I shall however only observe, that the friction of the pivots of each roller in the holes of the brass rings is very inconsiderable; for those holes are made rather large, the use of the axes to the rollers being only to prevent their running one against the other. Nor does the addition of weight upon the pulley increase that friction; for the addition of weight upon the pulley will press the rollers harder upon the axis A; but not upon their own axes, as may be easily understood by inspecting the figure.

CHAPTER XIV.

DESCRIPTION OF THE PRINCIPAL MACHINES,
WHICH ARE NECESSARY TO ILLUSTRATE THE
DOCTRINE OF MOTION, AND OF THEIR PAR-
TICULAR USE.

THE doctrine of motion in all its extensive branches is derived, as we have already shewn, from a few general principles; and its application to particular circumstances requires only the knowledge of a few experimental facts, such as the natural descent of bodies towards the earth, the time of vibration of a pendulum of a determinate length, &c. Then whatever relates to other complicated movements may be derived, by means of strict and unequivocal reasoning, from those few principles, and few ascertained facts or natural laws. Yet notwithstanding the assent which a rational being must give to the clear and evident demonstrations that are derived from those principles, it must be allowed that an experimental confirmation of any theoretical proposition never fails to impress the mind with a pleasing, lasting, and satisfactory conviction. Though the unavoidable imperfections of machines render the result of experiments seldom so accurate as to coincide exactly with

with the theory, yet when the error is not very great, and at the same time it seems to be proportionate to the imperfections of the mechanical construction and operation, the mind of the observer will always feel itself sufficiently satisfied.

In this chapter the reader will find the description of the principal machines which have been contrived for the purpose of confirming in an experimental manner the propositions which relate to motion.

The space described, and the acceleration gained, by bodies which descend freely towards the earth has been often attempted to be proved by means of direct experiments; but the resistance of the air which opposes a considerable and fluctuating impediment, and the difficulty of measuring the time of descent when falling bodies have acquired a great degree of velocity, which soon increases beyond the power of our senses to estimate, have always rendered the result of such experiments precarious and unsatisfactory. * But we are indebted to Mr.

* Dr. Desaguliers observed the time that a leaden ball, of two inches in diameter, employed in descending from rest through 272 feet; that is, from the inside of the cupola of St. Paul's cathedral, wherein the experiment was tried, to the floor; and found it to be 4,5 seconds, whereas it should have been 4,1 seconds; for in 4,5 seconds it ought to have descended through somewhat more than 325 feet. See his *Course of Experimental Philosophy*.

Atwood, F.R.S. for a very curious machine of his contrivance, which obviates the abovementioned impediments, and exhibits the phenomena of accelerated and retarded motion in a commodious and satisfactory manner; or, in Mr. Atwood's words, "which will subject to experimental examination the properties of the five mechanical quantities; that is, the quantity of matter moved, the constant force which moves it, the space described from rest, the time of description, and the velocity acquired."

The representation of this machine in fig. 3, Plate VIII. is divided into two parts for the convenience of the plate, which however can make no difference with respect to the explanation; for the reader needs only imagine that these two parts are placed one upon the other, and are joined at the places which are indicated by the same letters, EF GH, so as to form one entire figure.

The foot or pedestal of this machine is in the form of a cross, with adjusting screws, which serve to set the machine in a steady and perpendicular situation. A strong wooden pillar XGT, about five feet high, is firmly fixed upon the pedestal, and supports the wooden stage VD, which is secured upon it by means of the screw at D. Upon this stage there is another stage or stand, to which the wheel-apparatus is fixed. This apparatus consists of a brass wheel *a*, *b*, *c*, whose steel horizontal axis moves upon four friction-wheels; viz. one end of the axis rests

rests upon the circumferences of two friction-wheels, and the other end rests on the circumferences of the two others. The axes of the four friction-wheels are supported by, and move in, holes made in the brass frame which is fastened to the upper stage, and whose shape is sufficiently indicated by the figure.

There is a groove all round the circumference of the wheel *a, b, c*, for the reception of a fine flexible silk line, at the extremities of which the bodies *A, B* are suspended. By this means the motion of the wheel *a, b, c*, with the silk line is rendered so very free, that when the bodies *A* and *B* are equal, if one of them be gently impelled upwards or downwards, both bodies will readily move in contrary directions; the friction of the axis being almost entirely removed by the application of the friction-wheels.

KEL is a scale or rod, divided into inches and tenths; and is so situated that one of the bodies, viz. *A*, may move very near the surface of it. *C* and *I* are two little stages, either of which may be fixed, by means of the lateral screw *M* or *N*, on any part of the scale *LEK*. The former of those stages serves to stop the body *A*, when descending, at any required height. The latter stage has a perforation sufficiently large to permit the free passage of the body *A*; but its use is to support occasionally a weight in the form of a bar, like the one seen upon it.—The perspective representation of this stage *I*,
is

is considerably strained for the purpose of rendering its construction more intelligible.

Upon the pillar of this machine there is adapted a simple sort of time-piece, consisting of a pendulum which vibrates seconds, and is kept in motion for a few minutes by means of a wheel and weight O. On the axis of the wheel there is a hand or index, which indicates the number of seconds on the dial Z. The use of this time-piece is to shew by the beats of the pendulum, the time which is employed by the body A in ascending or descending through a given space *.

The useful property of this machine is to diminish the force which acts upon, and occasions the descent of bodies, in consequence of which a body will descend much slower; hence the observer will be enabled to perceive the space it moves through, as also its acceleration in a given time, &c. in a clear and commodious manner. I shall endeavour

* It hardly needs be observed that those who have a common clock, that beats seconds, may have the machine constructed without the last-described appendage. Besides this, I shall just mention that the abovementioned machine has been improved, or rather altered, by some philosophical instrument makers; but as those alterations are not of great importance either with respect to its construction or to its performance, I have preferred Mr. Atwood's original construction; such as is described in his very valuable treatise on the rectilinear motion, and rotation of bodies.

to render the explanation of this property more intelligible, previously to the narration of the experiments.

When the weights or bodies A and B are exactly equal, they will balance each other, and of course will remain at rest. But if a body of little weight be added to one of those bodies, as for instance to A, then A will preponderate, and consequently will descend; the opposite weight B ascending at the same time. Now in this case both the bodies A, B, and the wheels are put in motion by the gravity of the small additional body; so that the sum of all those bodies, being moved by a smaller force, must move through a shorter space in a given time, than if the force were greater.

For instance, imagine that the weights A and B, together with the weight which is required to put the wheels in motion (which is equivalent to the *inertia* of the wheels) amounts to 4 ounces, and let the weight of the body which is added to A be half an ounce, then it is evident that a mass of matter of $4\frac{1}{2}$ ounces is put in motion by the gravity of a body of half an ounce; that is by the gravity of a body equal to the 9th part of the matter which is put in motion, which amounts to the same thing as if that mass of matter were attracted by the earth with the ninth part of its ordinary attraction. But it has been shewn in page 64, that the space which is described in a given time by a descending body, is proportionate to the force of gravity;

gravity; therefore if in the natural way a body descends from rest through 16,087 feet in the first second of time, in the above-stated circumstance the body A will descend through the ninth part of 16,087 feet, viz. through 21,4 inches, in the first second of time. Thus by adding a smaller weight to the body A, that body may be made to move as slowly as the observer pleases.

The other properties of descending bodies remain unaltered by this machine. Thus the spaces which are described by the descending body A will be found to be as the squares of the times; that is, if A describe 21,4 inches in the first second of time, it will describe 4 times 21,4 inches in the second second of time, 9 times 21,4 inches in the third, &c. Thus much may suffice with respect to the principal effect of this machine. I shall now add Mr. Atwood's computation, and general mode of conducting the experiments.

In the first place he ascertained the inertia of the wheels when the silk line with the bodies A, B was removed, and found it equivalent to $2\frac{1}{4}$ ounces (1).

“ The

(1.) Having removed the weights A and B, with their silk line, Mr. A. affixed a weight of 30 grains to a silk line (the weight of which was not so much as $\frac{1}{4}$ of a grain, and consequently too inconsiderable to have any sensible effect in the experiment); this line being wound round the wheel *abc*, the weight of 30 grains, by descending from rest, communicated

“The resistance to motion, therefore, arising from the wheel’s inertia will be the same as if they were absolutely removed, and a mass of $2\frac{3}{4}$ ounces were uniformly accumulated in the circumference of the wheel *abc*. This being premised, let the boxes A and B be replaced, being suspended by the silk line over the wheel *abc*, and balancing each other.

communicated motion to the wheel, and by many trials it was observed to describe a space of about 38,5 inches in three seconds. From these data the inertia of the wheels may be determined in the following manner :

If the weight of 30 grains had descended through 9 times 193 inches in three seconds, as it would have done by itself, the inertia of the wheels would have amounted to nothing; but since it moved through 38,5 inches in three seconds, its retardation was occasioned by the inertia of the wheels. Let the quantity of this inertia be called x ; then the attractive force of the earth upon the mass $x+30$ must be less than upon the body of 30 grains alone; therefore $x+30$ descends slower than the body of 30 grains would by itself; or, properly speaking, the spaces which are described in the same time, are inversely as the masses; for the quantity of force being the same, the effect upon $x+30$ must be as much less than the effect upon 30, as 30 is less than $x+30$; hence in the present experiment the space described by the body of 30 grains in three seconds, is to the space described by $x+30$ in the same time, as $x+30:30$, viz. $9 \times 193:38,5 :: x+30:30$; therefore $x+30 = \frac{30 \times 9 \times 193}{38,5}$
 $= 1353,5$ grains, and $x = 1323,5$ grains, or $2\frac{1}{4}$ ounces.

“ To proceed in describing the construction of the ensuing experiments. In order to avoid troublesome computations in adjusting the quantities of matter moved, and the moving forces, some determinate weight of convenient magnitude may be assumed as a standard, to which all others are referred. This standard weight in the subsequent experiments is $\frac{1}{4}$ of an ounce, and is represented by the letter *m*. The inertia of the wheels being therefore equal to $2\frac{1}{4}$ ounces, will be denoted by $11m$. A and B are two boxes constructed so as to contain different quantities of matter, according as the experiment may require them to be varied: the weight of each box, including the hook to which it is suspended, is equal to $1\frac{1}{2}$ oz. or $6m$; these boxes contain such weights as are represented by Q, each of which weighs an ounce, or $4m$: other weights of $\frac{1}{2}$ an ounce $= 2m$, $\frac{1}{4} = m$, and aliquot parts of *m*, may also be included in the boxes, according to the conditions of the different experiments.

“ If $4\frac{1}{2}$ oz. or $19m$, be included in either box, this with the weight of the box itself will be $25m$; so that when the weights A and B, each being $25m$, are balanced in the manner above represented, their whole mass will be $50m$, which being added to the inertia of the wheels, $11m$, the sum will be $61m$. Moreover, three circular weights, such as that which is represented by Y, are constructed; each of which is equal to $\frac{1}{4}$ oz. or *m*: if one of these
be

be added to A and one to B, the whole mass will now become $63m$, perfectly in equilibrio, and moveable by the least weight added to either (setting aside the effects of friction) in the same manner precisely as if the same weight or force were applied to communicate motion to the mass $63m$, existing in free space and without gravity."

OF THE MOVING FORCE.

"Since the natural weight or gravity of any given substance is constant, and the exact quantity of it easily estimated, it will be convenient in the subsequent experiments to apply a weight to the mass A, as a moving force: thus when the system consists of a mass $= 63m$, according to the preceding description, the whole being perfectly balanced, let a weight of $\frac{1}{64}$ oz. or m , such as Y, be applied to the mass A, this will communicate motion to the whole system." But since now the whole mass is 64, and the moving force is the gravity of one of those parts only; "therefore the force which accelerates the descent of A, is $\frac{1}{64}$ part of the accelerating force by which bodies descend freely towards the earth's surface."

Thus by varying the weights, the moving force may be altered without altering the mass; or the moving force may be made to be in any required ratio to the mass.

OF THE SPACE DESCRIBED.

“The method of estimating practically the space described from quiescence, is next to be considered. The body A descends in a vertical line, and a scale of about 64 inches in length, graduated into inches and tenths of an inch, is adjusted vertically, and so placed that the descending weight A may fall in the middle of the square stage, fixed to receive it at the end of the descent; the beginning of the descent is estimated from o on the scale, when the bottom of the box A is on a level with o. The descent of A is terminated when the bottom of the box strikes the stage, which may be fixed at different distances from the point o, so that by altering the position of the stage, the space described from quiescence may be of any given magnitude less than 64 inches.”

CONCERNING THE TIME OF MOTION.

“The time of motion is observed by the beats of the pendulum which vibrates seconds: and the experiments intended to illustrate the elementary propositions may be easily so constructed, that the time of motion shall be a whole number of seconds; the estimation of the time therefore admits of considerable exactness, provided the observer take care to let the bottom of the box A begin its descent precisely at any beat of the pendulum; then

then the coincidence of the stroke of the box against the stage, and the beat of the pendulum at the end of the time of motion, will shew how nearly the experiment and the theory agree together. There might be various mechanical devices thought of for letting the weight A begin its descent at the instant of a beat of the pendulum; but the following method may perhaps be sufficient: let the bottom of the box A, when at o on the scale, rest on a flat rod held in the hand horizontally, its extremity being coincident with o; by attending to the beats of the pendulum, and with a little practice, the rod which supports the box A, may be removed at the instant the pendulum beats, so that the descent of A shall commence at the same instant."

OF THE VELOCITY ACQUIRED.

"It remains only to describe in what manner the velocity acquired by the descending weight A, at any given point of the space through which it has descended, is made evident to the senses. The velocity of A's descent being continually accelerated, will be the same in no two points of the space described: this is occasioned by the constant action of the moving force; and since the velocity of A at any instant is measured by the space which would be described by it, moving uniformly for a given time with the velocity it had acquired at

that instant, this measure cannot be experimentally obtained, except by removing the force by which the descending body's acceleration was caused."

"In order to shew in what manner this is effected practically, let us suppose that, according to a former example, the boxes A and $B = 25m$ each, so as together to be $= 50m$; this with the wheels inertia $11m$ will make $61m$: now let m be added to A , and an equal weight m to B , those bodies will balance each other, and the whole mass will be $63m$. If a weight m be added to A , motion will be communicated, the moving force being m , and the mass moved $64m$. In a former example, the circular weight, equal m , was made use of as a moving force; but for the present purpose of shewing the velocity acquired, it will be convenient to use a flat rod (like that which is shewn at I on the perforated stage) the weight of which is also equal to m . Let the bottom of the box A be placed on a level with o on the scale, the whole mass being as described above, $63m$, perfectly balanced in equilibrio. Now let the rod, the weight of which $= m$, be placed on the upper surface of A ; this body will descend along the scale precisely in the same manner as when the moving force m was applied in the form of a circular weight. Suppose the mass A to have descended by constant acceleration of the force m , for any given time, or through a given space: let the perforated stage be

be so affixed to the scale contiguous to which the weight descends, that *A* may pass centrally through it, and that this perforated stage may intercept the rod *m*, by which the body *A* has been accelerated from quiescence. After the moving force *m* has been intercepted at the end of the given space or time, there will be no force operating on any part of the system, which can either accelerate or retard its motion; this being the case, the weight *A*, the instant after *m* has been removed, must proceed uniformly with the velocity which it had acquired that instant: in the subsequent part of its descent, the velocity being uniform will be measured by the space described in any convenient number of seconds."

OF RETARDED MOTION.

"The motion of bodies resisted by constant forces are reduced to experiment by means of the instrument above described, with as great ease and precision as the properties of bodies uniformly accelerated. A single instance will be sufficient: thus suppose the mass contained in the weights *A* and *B*, and the wheels, to be $61m$, when perfectly in equilibrio, as in a former example; let a circular weight *m* be applied to *B*, and let two long weights or rods, each equal to *m*, be applied to *A*, then will *A* descend by the action of the moving force *m*, the mass moved being $64m$: suppose that when

it has described any given space by constant acceleration, the two rods m are intercepted by the perforated stage, while A is descending through it; the velocity acquired by that descent is known, and when the two rods are intercepted, the weight A will begin to move on with the velocity acquired, being now retarded by the constant force m ; and since the mass moved is $62m$, it follows, that the force of retardation will be $\frac{1}{62}$ part of the force whereby gravity retards bodies thrown perpendicularly upwards. The weight A will therefore proceed along the graduated scale in its descent with an uniformly retarded motion, and the spaces described, times of motion, and velocities destroyed by the resisting force, will be subject to the same measures as in the examples of accelerated motion above described."

Besides those properties, Mr. Atwood's machine may be easily adapted to other uses, such as the experimental estimation of the velocities communicated by the impact of bodies elastic and non-elastic; the quantity of resistance occasioned by fluids, &c. Mr. Atwood also shews its use in verifying practically the properties of rotatory motion*.

After the preceding sufficiently ample description of the general mode of using this instrument,

* See his Treatise on Motion, Sect. VIII.

we shall by way of example subjoin three or four experiments, and shall leave its further application to particular cases, for the exercise of the reader's ingenuity. It is however necessary, in the first place, to obviate some doubts which may naturally occur with respect to the performance of this machine; the accuracy of which may be disturbed by three causes, viz. the friction of the axes of the wheels, the weight of the silk line, and the resistance of the air.

“The effects of friction are almost wholly removed by the friction-wheels; for when the surfaces are well polished and free from dust, &c. if the weights A and B be balanced in perfect equilibrium, and the whole mass consists of 63*m*, according to the example already described, a weight of $1\frac{1}{2}$ grain, or at most 2 grains, being added either to A or B, will communicate motion to the whole, which shews that the effects of friction will not be so great as a weight of $1\frac{1}{2}$ or 2 grains. In some cases, however, especially in experiments relating to retarded motion, the effects of friction become sensible; but may be very readily and exactly removed by adding a small weight of $1\frac{1}{2}$ or 2 grains to the descending body, taking care that the weight added is such as is in the least degree smaller than that which is just sufficient to set the whole in motion, when A and B are equal, and balance each other, before the moving force is applied.”

The

The silk line by which the weights are suspended is 72 inches long, and weighs about 3 grains: a quantity too small to affect sensibly the result of the experiments.

The effect of the resistance of the air is likewise insensible; for that resistance increases with the velocity, and in the experiments which are performed with this machine, the greatest velocity communicated to the bodies A and B, cannot much exceed that of about 26 inches in a second.

Experiment 11th. Let A and B, together with the $2\frac{3}{4}$ oz. (which are equal to the inertia of the wheels), amount to 16 oz. or $63m$; then add a weight of $\frac{1}{4}$ oz. that is m , to A, and A will descend and will describe from rest three inches in the first second; so that if the square stage be fixed even with the 3 inches on the scale, and A be permitted to descend from 0 on the scale just when the pendulum strikes, it will be found that exactly when the pendulum strikes the next stroke, the body A will strike against the stage. If the experiment be repeated with this variation only, viz. with the stage fixed even with the 12 inches on the scale, then the weight A will strike the stage exactly when the pendulum strikes the second stroke after the commencement of A's motion. And if the stage be fixed even with the 27 inches, the stroke of A on the stage will coincide with the third stroke of the pendulum; and so on.

Here

Here it is evident that the quantity of matter in motion is represented by 64 parts, and this quantity is put in motion by the gravity of one of those parts; therefore the moving force being the 64th part of what the earth would otherwise exert upon the whole mass, this mass must move through the 64th part of that space which, if descending freely, it would move through in the same time. But in the natural way descending bodies pass through 16,087 feet, or 193 inches, in the first second; therefore in this experiment the body A must descend through the 64th part of 193, viz. 3 inches nearly. In two seconds it must descend through 4 times 3, or 12 inches; in three seconds it must descend through 9 times 3, or 27 inches, &c. the spaces being as the squares of the times.

Experiment 2d. If the weight of A and B, together with the inertia of the wheels, be made equal to $62m$, and a weight of $2m$ be added to A, then the whole mass in motion will be $64m$, and the moving force $2m$, viz. $\frac{1}{32}$ of the mass; therefore in the first second of time A will be found to descend through a space equal to the 32d part of 193, viz. 6 inches. In two seconds it will descend through four times 6, viz. 24 inches, and so on.—Thus the force may be varied at pleasure, and the space described by the descending body in a given time will be found proportionate to the force.

Experiment 3d. Let the quantity of matter be $63m$, as in the first experiment; add a bar of the weight m to A, and place the perforated stage even
with

with the 12 inches, and let the weight A commence its descent when its upper surface is even with the 0 on the scale. It will be found that in two seconds the bar on the body A will strike against, and remain on, the perforated stage, after which the body A, not being any longer acted upon by any accelerative force, will continue to descend with an equable motion, and will describe (according to the law which has been mentioned and proved in page 67.) a space equal to twice the above-mentioned descent in the same time, that is, 24 inches in 2 seconds. Thus the degree of velocity acquired after any other descent may be proved experimentally.

Experiment 4th. Let A be equal to $24\frac{1}{2}m$, and B equal $25\frac{1}{2}m$, and apply to the upper surface of A two rods, each of which is equal m , then will the weight A preponderate and descend by the action of a moving force equal to m ; the whole mass moved being equal to $63m$. Fix the perforated stage at 26,44; then the weight A by descending from rest through 26,44 inches, will acquire a velocity equal to 18 inches per second:

(viz. the square root of $\frac{4 \times 193 \times 26,44}{63}$) and at

that instant the two rods, each of which is equal to m , being intercepted by the stage, the body A will continue to descend with an uniformly retarded motion; which will be precisely the same as if a mass of $61m$, without gravity, were projected with

a ve-

a velocity of 18 inches in a second in free space, and a force or resistance equal to m were opposed to its motion; wherefore A (with the other parts of the system) will lose its motion gradually, and will describe a space equal to 25,6 inches (that is, $\frac{18 \times 18 \times 61}{4 \times 193}$) before its motion is entirely destroyed: A will therefore be observed in the experiment to descend as low as 52 inches, before it begins to ascend by the superior weight of B .

The next machine we shall describe is called a *whirling-table*, and its use is for shewing, in an experimental way, the nature and properties of centripetal and centrifugal forces.

The machine itself is exhibited by fig. 1, Plate IX. and the apparatus is represented by the numbers 1, 2, 3, &c. adjoining to it*.

Upon the steady table fff , the two strong pillars e, e are immoveably fixed, which are also steadily screwed to the cross piece ab . Within this frame the two upright hollow axes are situated so that each of them may turn with a pointed pin in a hole on the table, and with its upper extremity through a hole in the cross piece ab . The lower part of

* Whirling tables have been varied more or less in shape and size by almost all the different makers of those instruments. That which I have preferred has considerable advantage in point of simplicity and durability. This machine was contrived and made by Mr. J. B. Haas.

each axis is immoveably connected with a doubly grooved wheel or pulley KH, CY parallel to the table. The grooved wheel B turns also parallel to the table, round a strong pin or axis which is fixed to the table; and a catgut-string is disposed round the wheel B, and round the large or small circumference of the wheel at the bottom of each axis, in the manner which is clearly indicated by the figure. In this disposition it is easy to conceive that by applying the hand to the handle at A, and turning the wheel B, both the axes will be caused to turn round. A socket, or tube I, I is connected with a circular brass plate FG, ED, and slides freely up and down each axis. From the inside of each of those tubes or sockets a wire passes through an oblong slit, and projects within the cavity of the axis, where it is shaped like a hook; so that a string may be tied to this hook, which passing upwards through the aperture of the axis; may be pulled or let down in such a manner as to let the plate and socket move up or down the axis. Upon those plates, semicircular leaden weights *o o* may be placed occasionally.—Those weights, being perforated, are slipped over two wires which proceed from the plate ED, or FG; as they are indicated by the figure; by which means the weights are prevented from falling off.

To the upper part of each axis (*viz.* to the part of it which projects above the cross piece *ab*) a variety of different mechanisms may be occasionally

ally

ally screwed so fast as to turn with the axis when the machine is in action.

The oblong pieces which are represented in the figure as being actually fixed to the axes, are called *bearers*.—Their construction being exactly the same, we need describe only one of them.

A perforated brass plate with a strong screw which fits the screw at the top of the axis; is fixed in the middle of the bearer *ML*; so that when the bearer is screwed to the axis; the hole in it communicates with the cavity of the axis. On one side of this hole; a perpendicular projection *T* rises above the surface of the bearer, and a similar projection rises above the end *L* of the bearer; which is on the other side of the central hole. Two smooth, strong, and parallel wires are stretched between those two projections by means of the screw-nuts at *W*. A cylindrical heavy body *V* is perforated with two longitudinal holes, through which the abovementioned wires pass, so that the body may be freely moved backwards and forwards upon those wires. On that side of the cylinder *V*, which lies towards *T*, there is a hook, to which a string is fastened. This string passes through a hole in the projection *T*; after which it goes round the grooved pulley *S*, which moves round an axis in an upright frame *h*, fixed to the bearer, and whose situation is such that the string in its descent at *T*, may pass through the middle of the hole in the bearer, and of the cavity of the axis, so as to
be

be fastened with its extremity to the hook of the wire which proceeds from the socket of the plate E D. After this description it is easy to understand, that the cylinder V and the plate E D are connected together by means of the string, and that if V be drawn towards W, in which situation it appears in the figure, the plate E D with its superincumbent weights will be pulled towards the upper part of the axis; otherwise the weight of the plate E D will draw the cylinder V towards T, and will itself descend towards the lower part of the axis.

Either of those bearers may be removed from, and one of the following mechanisms may be screwed fast upon, the axis.

No. 1. represents a circular board, turned upside down, having a strong screw in its middle, which fits the screw at the top of either axis of the machine. There is a hole through the middle of this board and of its screw, which opens the communication with the cavity of the axis. But this hole in the middle of the board may be occasionally filled up with a piece of wood in the form of a stopple, which is furnished with a short pin, that, when the piece of wood is fixed in the hole, projects a little above the surface of the board.

No. 2. is an oblong bearer, which may be screwed, like any of the others, upon one of the axes of the machine. It has an upright projection at each end, and a strong and smooth wire is stretched be-

tween

tween those projections by means of the screw nuts, A, B. C and D are two perforated brass balls, of unequal weights, which are connected together by means of a brass tube, and are freely moveable upon the wire AB from one end to the other. On the outside of the brass tube which connects the two balls, there is fixed, exactly at the common centre of gravity of those balls, a short wire E as an index, which serves to shew when the common centre of gravity of those balls is placed exactly against the middle of the bearer.

No. 3. represents a board having at its lower end C, a screw which fits the screw at the top of one of the axes of the machine, upon which it may be firmly screwed; but this screw is situated a little aslant to the board, so that when placed upon the axis of the machine this board may stand inclined to the horizon, making an angle of 30 or 40 degrees with it.

On the upper side of this board are fixed two glass tubes, AG and BF, close stopp'd at both ends; and each tube is about three-quarters full of water. In the tube BF is a little quicksilver, which, in consequence of its weight, remains under the water at the end B. In the other tube AG is a piece of cork, which being lighter than water, floats upon it towards the end G, and is so small as not to stick fast within the cavity of the tube.

No. 4. is an axis or strong wire fixed to a board, and having a screw at its lower part beneath the
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board, which fits the screw-hole at the top of one of the axes of the whirling table. Two circular brass hoops, AC, BD, made very thin and pliable, foldered to each other, and foldered or screwed to the axis, at I, have each a hole at the upper part through which the axis passes freely ; so that if a hand be applied to the upper part E of those hoops, they may be flattened down as far as the pin O, which is seen across the axis. In this case the hoops will change their circular form into an elliptical one ; but, being elastic, they will resume their circular form as soon as the pressure is removed.

No. 5. represents a hemisphere, which is to be situated upon the board No. 1, when that is fixed upon one of the axes of the machine, in the following manner : A pin with a screw *e* (which is not fixed to the hemisphere) is screwed in the middle of the board so as to project a little above it, and the hemisphere A is laid upon it, there being a cavity *d i* on the flat part of the hemisphere made on purpose to lodge the pin ; but this cavity is an oblong groove, as is pretty well indicated by the dotted line on the figure of the hemisphere ; and it is made so that the hemisphere by sliding over the pin the whole length of that groove, may be placed either concentric with the board, or out of centre with it. The lateral wire C, with the small ball B, may be screwed occasionally on the
side

side of the hemisphere A, in a direction opposite to that of the abovementioned groove.

No. 6. represents a forked wire with a screw at its lower part, which fits the screw in the middle of the board No. 1. This fork serves to support the wire C with the two unequal balls A and B; but this wire being in no way connected with the forked wire, must be balanced upon it, that is, it must be laid with the common centre of gravity upon the fork; which is easily done by trial.

No. 7. represents a ball of about two inches in diameter, having a hole from side to side, through which a wire BA passes quite freely. This wire out of the ball at A is shaped like the letter T, each of whose projections is longer than the radius of the ball, and has a blunt termination. On the other side B the wire terminates in a ring, to which a string is tied.—We shall now proceed to describe the experiments which are to be made with the whirling table and its apparatus.

Experiment 1. Fix the board No. 1. upon one of the axes of the machine, and put the piece of wood or stopper with the pin, in the middle of it. Take the ball apparatus, No. 7, make a loop on the end C of the string, taking care that the length CA be not greater than the radius of the board. Put the loop of the string over the pin in the middle of the board, and leave the ball upon the board. Then apply a hand at A, and turn the wheel B of the machine, which will give the board

a whirling motion. It will be found that the ball does not immediately begin to move with the board; but, on account of its *inertia*, it endeavours to continue in its state of rest, in which it stood before the machine was put in motion. But, by means of the friction on the board, that *inertia* is gradually overcome; so that by continuing to whirl the board, the ball's motion will become equal to that of the board; after which the ball will remain upon the same part of the board, it being then relatively at rest upon the board. But if you stop the board suddenly, by applying a hand to it, the ball will be found to go on in virtue of its *inertia*, and continue to revolve, until the friction of the board, by gradually diminishing its velocity, finally stops it. This shews that matter is as incapable of stopping itself when in motion, as it is incapable of moving itself from a state of rest.

Experiment 2. Remove the piece of wood with the pin from the middle of the board. Instead of the string with the loop, put a longer string to the ring B of the ball No. 7. Let this string down through the hole in the middle of the board, and through the cavity of the axis, and fasten it to the ring of the wire which proceeds from the socket of the plate FG; the weight of which will draw the ball towards the centre of the board. Care must be had to let the string be of such a length as that when the plate FG is quite down, the ring
B of

B of No. 7. be about an inch from the hole in the middle of the circular board. The weight of the plate FG must be very little more than sufficient to draw the ball to the abovementioned situation; for which purpose the leaden weights must be removed from over the plate FG, its own weight alone being sufficient for the purpose.

Having placed the ball so that the ring B may be about one inch distant from the centre of the board, put the machine in motion by turning the wheel B; and it will be found that the ball by going round and round with the board, will gradually fly off to a greater and greater distance from the centre of the board, raising up the plate FG at the same time; which shews that all bodies which revolve in circles, have a tendency to fly off, so that a certain power from the centre must act upon them in order to prevent their flying off. If the machine be stopped suddenly, the ball will continue to revolve for some time longer; but the friction of the board gradually diminishing its velocity, its tendency to fly off will also decrease, and the weight of the plate FG will gradually draw it nearer and nearer the centre, until its motion ceases entirely.

Experiment 3d. Let the apparatus remain as in the preceding experiment, excepting only that the string, being disengaged from the plate FG must be let out of the slit in the axis, and the operator must hold its extremity in his hand. With

his other hand the operator must throw the ball upon the circular board as it were in a direction perpendicular to the string, by which means the ball will make several revolutions upon the board (the machine being in this experiment at rest). But if whilst the ball is revolving you gradually pull the lower end of the string below the board, you will find that the ball, in proportion as it comes nearer to the centre of motion, and of course it performs its revolutions in smaller circles, will revolve faster; which shews, as far as such a machine can do it, that the same moving force will enable a revolving body to describe a circular orbit faster when the circle is smaller, and slower when the circle is larger. (See chap. VIII.)

Experiment 4th. Remove the circular board, and instead of it, put the bearer on the axis; so that both the bearers may be upon the machine, as is represented in the figure.

Let the cylinders R V, be of equal weights; place equal weights upon the plates FG, ED; and adjust the lengths of the strings which connect those cylinders with those plates, so that when the plates are quite down the cylinders may stand at equal but small distances from the centres of their respective bearers. The catgut string must be put either over both the large, or over both the small circular grooves at the bottom of the axes (on which account it is necessary to have two catgut strings, viz. one longer than the other) then put the
 4 machine

machine in motion, and the cylinders R, V will be seen to recede from the centres of the bearers, and to advance towards the ends X and W, raising at the same time, and to an equal height, the plates FG, ED. This experiment proves that when equal bodies revolve in equal circles, with equal velocities, their centrifugal forces are equal.

Experiment 5th. Instead of the cylinder R, place another cylinder of half its weight, viz. equal to half the weight of V, on the wires of the bearer PN; adjust the strings so that when the plates FG, ED are quite down, the distance of the cylinder V from the centre of the bearer ML may be half the distance of the other cylinder from the centre of its bearer, which is easily shewn by the divisions which are marked upon the bearers; and leave the rest of the apparatus as in the preceding experiment. Now when the machine is put in motion, there will be two bodies revolving, one of which is half the weight of the other, but the former revolves in a circle which is as large again as the circle in which that other revolves. And it will be found that the equal weights of the plates FG, ED will be equally raised; which shews that the centrifugal forces of the revolving bodies (which raise the plates FG ED) are equal as long as the products of the bodies multiplied each by its velocity, viz. the momentums, are equal.

The proportion of the weights of the two bodies may be varied at pleasure; but the strings must be

adjusted so that their distances from the centres of the bearers may be inversely as those weights; and the plates FG, ED, which are loaded with equal weights, will always be lifted to equal heights; the products of the bodies by their respective velocities being always equal.

Experiment 6th. Repeat the preceding experiment, with this difference, that the cylinders be left at equal distances from the centres of their respective bearers; also that the weights on the plates FG, ED be in the proportion of the weights of the cylinders, viz. when V weighs as much again as R, the weight of the plate ED must be double the weight of the plate FG, &c. On putting the machine in motion it will be found that the plates FG, ED are raised to the same height; which proves that when revolving bodies move with the same velocity, their centrifugal forces are proportionate to their respective quantities of matter.

Experiment 7th. Put cylinders of equal weights on the wires of the bearers PN, ML. Place the catgut string round the wheel B, the wheel Y, and round the small wheel H, which is exactly the disposition represented by the figure. Also, adjust the strings between the cylinders and the axes, so that when the plates FG, ED are quite down, the cylinders may lie at equal distances from the centres of the bearers. Farther, if the circumference of the wheel Y be equal to twice that of the wheel H, you must put four times as much weight on

the plate FG, as upon the plate ED. If the circumference of the wheel Y be equal to three times that of the wheel H, you must put nine times as much weight on FG as upon ED. In short the weights on the plates must be inversely as the squares of the circumferences of the wheels Y and H. On putting the machine in motion it will be found that the plates FG, ED are raised to an equal height; which shews that when equal bodies revolve in equal circles with unequal velocities, their centrifugal forces are as the squares of the velocities.

Experiment 3th. Let the catgut remain in the same situation as in the last experiment, and let the circumference of the wheel Y be to that of the wheel H as two to one (in which proportion the circumferences of those wheels of whirling machines are generally constructed). Also let the cylinders R and V be of equal weights, but adjust the strings so that when the plates FG, ED are quite down, the distance of the cylinder R from the centre of the bearer PN be two inches, whilst that of the cylinder V from the centre of the bearer ML be $3\frac{1}{6}$ inches*. The circumference of the wheel Y being equal to twice

* Instead of inches, the distances may be of any other denomination; provided they be in that proportion. The bearers are generally divided into equal parts which are longer than inches; so that the distances may be made equal to 2 and $3\frac{1}{6}$ of those divisions.

the circumference of the wheel H, it follows that when the machine is put in motion, V must make one revolution whilst R makes two; therefore their periodical times are as one to two, and the squares of those times are 1 and 4; the former of which is contained four times in the latter. But the distance of R is 2, the cube of which is 8; and the distance of V is $3\frac{1}{2}$, the cube of which is 32 nearly, in which 8 is contained 4 times; therefore the squares of the periodical times are as the cubes of the distances. Now let the weight of the plate ED be 4 ounces, equal to the square of the distance 2; and the weight of the plate FG be 10 ounces, nearly equal to the square of the distance $3\frac{1}{2}$; then on turning the wheel B, which will put the axes in motion, it will be found that the plates FG, ED are raised to an equal height.

This experiment proves that when equal bodies revolve in unequal circles, and the squares of the times of their going round are as the cubes of their distances from the centres of the circles, then their centrifugal forces are inversely as the squares of their distances.

Experiment 9th. Remove one of the bearers from the machine, and place the mechanism No. 3, upon the axis. (See the description of this mechanism in p. 305.) On turning the wheel B, it will be found that the contents of the glass tubes AG, BF will, in consequence of their centrifugal forces, run towards the outward and uppermost ends
of

of those tubes. And since with equal velocities the heaviest bodies have the greatest centrifugal force, therefore the quicksilver in the tube BF will go quite to the end F of the tube; its weight being greater than that of an equal bulk of water; but in the tube AG the piece of cork will be found at the bottom of the water; the water being much heavier than an equal bulk of cork.

Experiment 10th. Remove the apparatus No. 3, and place No. 4. upon the axis. On putting the machine in motion, the upper part E of the hoops will descend towards the pin O, and the quicker the machine is whirled, the nearer will the hoops come to the pin, their middle parts receding at the same time from the axis; so as to assume an elliptical form. This effect arises from the different centrifugal forces of the different parts of those hoops; the centrifugal forces of those parts which are farther from the axis of motion being greater than of those which are nearer to it.

It appears therefore, that when globular bodies, whose matter is sufficiently yielding, revolve round their axes, their figure cannot be perfectly spherical, but it is that of an *oblate spheroid*.

Experiment 11th. Remove the preceding apparatus from the axis of the whirling table; place the board No. 1. upon it; fix the pin *e* of the machine No. 5. in the middle of the board, with the hemisphere A upon it, but without the wire C. If the hemisphere A be placed concentric with the board,

board, on whirling the machine, the hemisphere will be found to remain in its place upon the board; the centre of gravity of the hemisphere coinciding with the centre of motion. But if the centre of the hemisphere be placed a little on one side of the centre of motion, then on whirling the machine, the larger portion of the hemisphere, which lies on one side of the centre of motion, will acquire a greater centrifugal force, and consequently will draw the hemisphere that way; so that the pin sliding through the groove *di*, the hemisphere will at last be found with the part *d* upon the pin.

If the wire *C* with the little body *B* be screwed on the side of the hemisphere, and the latter be placed upon the pin, concentric with the board; on whirling the machine, the same thing as the last mentioned effect will take place; for though the hemisphere be placed concentric with the board, yet when the body *B* is affixed to it, their common centre of gravity is different from the centre of gravity of the hemisphere alone.

Experiment 12th. Nearly the same thing is shewn by means of the apparatus No. 2. For when this is screwed upon the axis of the whirling table, (the preceding mechanism being removed) if the index *E*, viz. the centre of gravity of the two bodies *C, D*, be placed exactly over the centre of the bearer *AB*, the whirling of the machine will not move the said bodies upon the wire *AB*; but if the centre of gravity *E* be placed ever so
little

little on one side of the centre of motion, on whirling the machine, the two bodies will move towards that side, as far as the upright projection A or B.

Experiment 13th. The same thing may be shewn by means of the mechanism No. 6. Place the circular board No. 1. upon one of the axes of the whirling machine; fix the forked wire D, of No. 6. in the middle of it; balance the bodies A,B, with their connecting wire, upon the fork; then put the machine in motion by turning the wheel B, and the bodies A,B will remain balanced upon the fork and will turn with it.

A vast variety of machines have been invented for the purpose of illustrating other branches of the doctrine of motion and equilibrium; but as the propositions which relate to such other branches are very easy and evident, the particular description of those machines would render the work voluminous, without proving of much advantage to the reader. I shall therefore only add a short account of the manner of shewing, by means of pendulums, the principal phenomena which attend the collision of bodies; and the description of a machine which serves to shew a few of the cases, which relate to the composition and resolution of forces, with which this chapter will be concluded.

The phenomena which attend the direct collision of bodies, viz. when their centres of gravity lie in the direction of their motion, may be very commodiously exhibited by means of pendulums, such

are represented in the figures 4, 5, 6 of Plate VIII; for if one of the pendulums as A in fig. 5, be removed to a certain distance from the perpendicular, as to the situation BC, and be then let go, the impulse which its ball gives to the next pendulum D, will force the latter to move from its state of rest, and to describe a certain arch, which will be longer or shorter according to the quantity of matter of the body which is struck, and according to the momentum of the striking body, which momentum may be increased or diminished by elevating the striking body to a greater or lesser angle, and by varying its weight. The pendulums may also be made to strike against each other after having been both put in motion, either the same way or contrary ways.

The effects of the collision, viz. the directions of the bodies after the stroke, and their velocities, may be estimated by observing the arches which they describe after the impact.

In this manner the experiments may be performed on elastic, as well as non-elastic, bodies. When the bodies are required to be elastic, ivory balls are suspended to the threads; but when the bodies are to be non-elastic, the balls are made of soft wax*, or of moist clay. And though the former be

* White wax may be rendered sufficiently soft for this purpose by melting it over a gentle fire and incorporating it with about one quarter of its weight of olive-oil; it may afterwards,

be not perfectly elastic, nor the latter perfectly non-elastic; yet the difference which arises from their imperfect properties, is so trifling, that it may be safely neglected in these experiments. And here it is proper to observe that in performing such mechanical experiments, wherein some allowance must be made on account of friction, of resistance of the air, of imperfect elasticity, &c. the result must be reckoned conclusive as long as the effect is somewhat less than what it ought to be according to the theory; but if the effect is greater than that which is determined by calculation; then some defect in the machinery, or error in the theory, must be suspected.

In the abovementioned figures the threads of the pendulums are represented as being fixed to common nails; but a machine, or stand, may be easily contrived (and many machines of this sort are described in almost all the books of mechanical philosophy*) upon which two or more pendulums may be easily suspended; where the lengths of the pendulums might be accurately adjusted, and where a graduated arch, as in fig. 5, might be easily applied,

afterwards, when cooled, be easily formed into balls, and the figure of the balls may be easily restored, after being altered in the course of the experiments.

* The best description of the construction and use of such a machine, is, in my opinion, that which is given at large in the second Book of s'Gravesande's *Mat. Elem. of Nat. Phil.* edited by Desaguliers.

for

for the purpose of measuring the arches from which the pendulums are permitted to descend, or those to which they ascend.

The facility with which such machines may be contrived and constructed renders the particular description of any of them in this place superfluous; one particular mechanism concerning it is however deserving of notice; and such is represented by fig. 7, Plate VIII. When a pendulous body, suspended by a single string, is raised to a certain height in order to give it motion, the least jerk or irregularity of the hand is sufficient to make it deviate from the proper plane of vibration, in which case the stroke on the other pendulous body will not be given in the direction of its centre of gravity; hence the effect will not turn out conformable to the theory. Now the suspension which is represented in fig. 7, avoids the possibility of that deviation; and therefore such suspension has been generally adopted for experiments of the abovementioned nature. DE is a slip of brass, the form of which is sufficiently indicated by the figure. It is fastened to the ball by means of a screw, and the thread BDEC, whose two extremities are fastened at B and C to a bracket, or horizontal arm of the machine, passes through two holes in the projections D, E of the brass slip.—It is evident that this pendulum must vibrate in a plane perpendicular to the plane BD EC of the figure, without any possible deviation.

Fig. 5. represents the case when the bodies are equal, and with this apparatus one of the bodies may be made to strike against the other at rest; or they may be made to strike against each other when they are both in motion. The same variety of experiments may be performed with the pendulums fig. 4. but in these the quantities of matter are unequal.

Fig. 6. represents the case in which three equal elastic bodies lie contiguous to each other, where if one of the outer bodies, as F, be lifted up to G, and then be permitted to descend against E, the stroke will be communicated from E to D; so that E will remain at rest, and D will be impelled up to H.—For the various cases of collision which may be exhibited by means of pendulums, see chap. VII.

Various machines have been contrived for the purpose of illustrating the composition and resolution of forces; and the weights sustained by oblique powers*. One of the clearest methods of shewing the composition of forces is the following:

Suspend two pendulums ACI, BD, as represented by fig. 8. Plate VIII. so that their balls may be very little above the surface of a flat and smooth

* Such machines are particularly described in most of the works on mechanics and natural philosophy, especially Gravesande's *Elem. of Natural Phil.* and Musschenbroek's *Philosophy*.

table LM RK. Place an ivory ball, E, upon the table, in contact with both the balls of the pendulums; then if you draw the pendulum A a certain way out of the perpendicular direction, and then let it fall against the ball E, in the direction EO, the ball E will be forced to move from E to O.— Replace the ball E in its former situation; raise the pendulum B so as to make an angle with the perpendicular, equal to that made by the other pendulum, then let it fall upon the ball E, whereby this ball will be forced to move from E to H. Lastly, put the ball E once more in its former situation; raise both the pendulums at the same time, and to the same angle to which they were before raised separately; then let them go both at the same instant, so that they may both strike the ball E at the same time; and the ball E will thereby be forced to move straight from E to G, which is the diagonal of the parallelogram GHEO, whose sides are the directions of the separate impulses EO, and EH.

The effect may be varied by increasing or diminishing one of the impelling forces, which must be done by increasing or diminishing the weight of one of the balls A or B.

In the use of such a machine, care must be had to let the two pendulums strike the ball E at the very same instant; which requires a considerable degree of dexterity. Mechanical means might indeed be easily contrived for the purpose of discharging

charging both the pendulums at the proper time ; but it is hardly worth while to construct a complicated machine for illustrating so evident a proposition.—In this machine the two pendulums are suspended nearly after the manner of fig. 7, viz. each pendulum is suspended by two threads, but the slip of brass is omitted.

CHAPTER XV.

CONTAINING THE APPLICATION OF SOME PARTS
OF THE FOREGOING DOCTRINE OF MOTION ;
WITH REMARKS ON THE CONSTRUCTION OF
WHEEL CARRIAGES.

OF all the branches of mechanics the properties of the centre of gravity occur most frequently, and are of the greatest consequence, to the human being.

Whatever body rests upon another body must have its centre of gravity supported by that other body, viz. the line drawn from its centre of gravity straight to the centre of the earth ; or, which is the same thing, the line which falls from its centre of gravity perpendicularly to the horizon, must be intercepted by, or fall upon, the other body ; otherwise the former will not be supported by the latter.

The abovementioned line, that is, a line drawn from the centre of gravity of a body, or system of bodies that are connected together, perpendicularly to the plane of the horizon, is called the *line of direction*; it being in fact the line along which the body will direct its course in its descent towards the centre of the earth; and, of course, in order to be supported, it must meet with an obstacle in that line. Thus in fig. 9. Plate VIII. the body CDOG will rest very well with its base upon the ground or other horizontal plane, because its line of direction IF, drawn from its centre of gravity I, perpendicular to the plane of the horizon, falls within the base YGO, every point of which is supported by the ground; but if another body ABCD be laid upon it, the whole will fall to the ground, for in the latter case the centre of gravity of the whole will be higher up, as at K, and the line of direction KH falls out of the base. Thus also in fig. 10. Plate VIII. the body D will roll down the inclined plane AB, because its line of direction falls without its base; whereas the body C, whose line of direction falls within its base, will only slide down that plane, unless the friction prevents it, in which case it will remain at rest; but friction will not prevent the rolling down of the body D.

It is therefore evident that the narrower the base is, the easier a body may be moved, and, on the contrary, the broader the base is, and the nearer the line of direction is to the middle of it, the more

firmly

firmly does the body stand. Hence it appears that a ball, or a circular plane figure standing upright, such as a wheel, is moved upon a plane with greater facility than any other figure; for the least change of position is sufficient to remove the line of direction of a spherical or circular body, out of the base. Hence also it is that bodies with narrow terminations, such as an egg or a stick, &c. cannot be made to stand upright upon a plane, at least not without the utmost difficulty.

The application of the properties of the centre of gravity to animal œconomy is easy and evident. If the line of direction falls within the base of our feet, we remain erect; and the steadiest, when that line falls in the middle of that base; otherwise we instantly fall to the ground.

On account of the great importance which the preservation and management of that centre is to animal motion, the infinite wisdom of the Creator has implanted in all animals a natural propensity to balance themselves in almost every circumstance. Many animals acquire the habit of keeping themselves upon their legs within a few hours after their birth, and such is particularly the case with calves.

It is wonderful to reflect, and to observe, how a child begins to try and improve his stability. He generally places his feet at a considerable distance from each other, by which means he enlarges the base, and diminishes the danger of a lateral fall:—he endeavours to stand quite erect, and with his body

immovable, so as to prevent as much as possible a fall backwards or forwards;—and when he lifts up one foot, he instantly replaces it upon the ground, finding his inability to rest upon so small a base as that of one foot. Farther advanced in years, he adopts farther methods of preserving and using the centre of gravity, and that without the least knowledge of the mechanical principle upon which he acts. Thus a man naturally bends his body when he rises from a chair, in order to throw the centre of gravity forwards. He leans forwards when he carries a burden on his back, in order to let the line of direction (which in that case descends from the common centre of gravity of his body and burden) fall within the base of his feet. For the same reason he leans backwards when he carries a burden before him; and leans on one side when he carries something heavy on his opposite side.

Human art improved by constant exercise and experience, goes far beyond those common uses of the centre of gravity, and line of direction. We see, for instance, men who can balance themselves so well as to remain erect with one foot upon a very narrow stand, or upon a rope, and even with their heads downwards and their feet uppermost.—Their art entirely consists in quickly counterpoising their bodies, the moment that the line of direction begins to go out of the narrow basis upon which they rest. Thus, if they find them-

selves

selves falling towards the right, they stretch out the left arm or the left leg, and *vice versa*; for though the weight of the arm be much less than the weight of the body; yet by being extended farther from the fulcrum, its momentum may be rendered equal to that of the rest of the body, which lies vastly nearer to the fulcrum, or to the line of direction.—*See chap. VI.*

This explanation likewise shows the great use of a long horizontal pole in the hands of a rope-dancer; for as the extremities of the pole, which are generally loaded with leaden weights, lie far from the rope, which is the fulcrum; when the pole is moved a little one way, the momentum of that extremity of it which lies that way, is increased considerably, and so as to counterpoise the body of the man, when he finds himself going the other way.

Notwithstanding the use of the centre of gravity which mankind acquires naturally or merely by experience; yet in many cases people are seen to act contrary to the laws of nature; and the consequences are sometimes quite fatal. Thus we frequently find that when a boat or carriage is over-setting, the persons in it rise suddenly from their seats; by which means they remove the centre of gravity of the whole higher up, and thereby accelerate the fall, (exactly like the case which has been represented in fig. 9, Plate viii.) which they might probably prevent, either by remaining on

their seats, or rather by lowering themselves down as close as they could to the bottom of the boat or carriage.

The natural application of the mechanical laws might be instanced in almost every occurrence of life ; for whatever moves must move conformably to those laws. But, to avoid prolixity, we shall only mention a few more particular instances : whence the attentive reader may easily learn how to apply the foregoing doctrine of motion to a variety of occurrences.

The man, or the horse, that runs in a circular path, naturally leans towards the centre of the circle, or towards the concave part of the curvilinear pathway ; and that in order to counteract the effect of the centrifugal force, which would otherwise throw him out of the perpendicular. And the swifter he runs, the more he leans towards the concave side ; the centrifugal force encreasing with the velocity.

When a man is to hold a great weight in his hand, he naturally places the hand near the body ; for if he extend his arm, the momentum of the weight which is placed at the end of it, as if it were at the end of a long lever, becomes too great for his power ; considering that the arm becomes a lever where the power and the fulcrum lie near one extremity, viz. near the shoulder, and the weight lies at the opposite extremity.

Persons

Persons who are accustomed to use a hammer generally hold it by the lowest part of the handle, for the purpose of rendering the stroke as powerful as possible; for in that case the head of the hammer, by being farthest from the centre of motion, moves with the greatest velocity, and, of course, strikes with the greatest momentum. But such persons as are not sufficiently accustomed to use a hammer, generally place their hand near the head of the instrument, by which means they render the stroke of very little effect.

The like observation might be made with respect to the use of almost all other tools and instruments, including those which are commonly in use, such as scissars, knives, razors, &c. And it is by the different management of such instruments that a mechanical hand is distinguished from an unmechanical or clumsy one; and that a person possessed of useful experience, useful habits and useful knowledge, is distinguished from one of the contrary description; excepting indeed when the awkward management of tools, &c. is wilfully adopted, under the refined idea, and for the purpose of shewing, that a person has never been under the disgraceful necessity of handling any mechanical instrument.

Of the different machines of luxury or convenience, that are in use amongst civilized nations, none have been more generally adopted, and more universally used, than wheel carriages; and yet it

is very remarkable, that, notwithstanding the present greatly improved state of mechanical knowledge, those machines are by no means constructed or used in the most advantageous manner possible.

Could the surfaces of bodies be rendered perfectly smooth, flat, and destitute of adhesion, it would be as easy to drag a body upon a plane, as it would be to move it upon wheels; but as this is far from being the case, the advantage which arises from the use of wheel carriages, is too evident to need any particular demonstration; and, in fact, we almost every day observe, that a single horse is able to carry upon a cart such a load as ten horses would perhaps have not strength sufficient to move on the bare ground.

When a heavy body is dragged upon the ground, the friction is very great, because the ground stands still and the body moves upon it, so that all the inequalities of the ground, the accumulation of dirt, and stones, the sinking of the ground, &c. form so many obstacles to the moving body, which obstacles must be overcome by the power which is applied to draw it. But when the body is carried upon wheels, as upon a cart, or waggon, &c. the surface of the rims of the wheels does not rub on, but is successively applied to the ground, (agreeably to what we have said above with respect to rollers; see page 279) and the obstacles arising from sinking, from stones, sand, &c. offer an oblique

lique opposition to the wheels, which is overcome vastly easier than a direct opposition, and the more so the larger the wheels are in diameter; so that by the use of large wheels the friction against the ground is almost entirely removed. There is however another sort of friction introduced by the use of wheels, viz. the friction of the wheel upon the axle; but this friction, when the parts are properly shaped, and oiled or greased, is not very material, especially when the wheels are large; for when a wheel turns upon an axle, the force necessary to overcome the friction is diminished in the ratio of the diameter of the wheel to the diameter of the axle.

A wheel carriage is drawn with the least power, when the line of draught passes through the centre of gravity of the carriage, and in a direction parallel to the plane on which it moves. It therefore follows that the height of the carriage should be regulated by the nature of the power which is to draw the carriage; viz. whether it is to be drawn by high or low horses, by bullocks, &c. It must however be observed, that, from the make of his body, a horse draws with the greatest advantage when the traces, or the shaft, makes a small angle with the plane which passes through the axles of the wheels. But this angle must not exceed a few degrees, otherwise part of the power of the horse is employed in lifting up the fore wheels from the ground;

ground; consequently the power is not entirely employed in drawing the carriage forward.

Four-wheeled carriages are almost always made with the two fore wheels smaller than the hind wheels. The fore wheels are made smaller than the others for the conveniency of turning, as they require less room for that purpose. The smallness of their size does also prevent their rubbing against the traces: but, those objects excepted, small wheels are by no means so advantageous as those of a larger diameter, as has been already mentioned, and as will be confirmed by the following illustration.

In fig. 2. Plate IX. let the hollows BGC, and DFO be equally large, and equally deep in the ground. It is evident that the large wheel A will not go so far into the hollowing, as the small wheel R. Besides, even supposing that they descend equally deep into those hollowings, the large wheel, by the power acting far above the impediment, may be easily drawn out of it; whereas the small wheel can hardly be drawn out by means of an horizontal draught, unless indeed when the ground gives way before it, which is not always to be expected.

The idea of the two large wheels helping to drive the fore small ones, is a vulgar error, which has not the least foundation in truth. The absurdity of this idea might be proved various ways, but by none more satisfactorily than by the following experiment.

Take

Take a real carriage, or the model of a large one, having two large and two small wheels. Fasten a rope at each of its ends, but equally high from the ground; then extending one of those ropes horizontally, let it go over a pulley, which must be placed at some distance from the carriage, and tie as much weight to the descending extremity of the rope, as may be just sufficient to move the carriage. This done, discharge this rope; turn the carriage with its other end towards the pulley, and, in short, repeat the experiment with the other end of the carriage foremost. It will be found that precisely the same weight will be required to draw the carriage, and to draw it with equal velocity, whether the large or the small wheels be placed foremost.

The figure, or rather the breadth of the rims of the wheels, influences considerably the motion of the carriage. Upon a smooth and hard road no advantage is derived from the use of broad wheels; but upon a soft road the broad wheels are much more advantageous than narrow ones; the latter cutting and sinking into the ground; on which account they must be considered as always going up hill, besides their suffering a great deal of friction against the sides of the ruts that are made by themselves; whereas the broad wheels produce nearly the same effect as a garden-roller; that is, they smooth and harden the road, besides their moving with great freedom. It must however be observed, that upon sand, as also upon stiff clayey roads,

roads, less force is required to draw a cart with narrow, than one with broad, wheels. Upon sand the broad wheels form their own obstacles, by driving and accumulating the sand before them. Upon clayey roads they gather up the clay upon their surfaces, and become in a great measure clogged by it.

Some persons imagine that the broad wheels, by touching the ground in a great many more points than narrow wheels, must meet with proportionably greater obstruction. But it should be considered, that though the broad wheels touch the ground with a larger surface, yet they press upon it no more than narrow wheels do. Let, for instance, two carts be equal in every respect and equally loaded, excepting that the wheels of one of them be 3 inches broad, whilst those of the other be 12 inches in breadth. It is evident that the latter wheels rest upon the ground with a surface which is equal to four times the surface upon which the former wheels rest. But since an equal weight is supported by the wheels of both carts, every three inches breadth on the surface of the broad wheels sustains a quarter of that weight; whereas the three inches breadth of the narrow wheels sustain the whole weight; so that the broad wheels touch the ground as much lighter as they are broader than the narrow wheels. It is for the same reason that if a heavy body in the form of a parallelepipedon, viz. like a brick, be dragged upon

a plane

a plane surface, the same power will be required to draw it along, whether its broad or its narrow side be laid upon the plane.

“ If the wheels were always to go upon smooth
“ and level ground, the best way would be to
“ make the spokes perpendicular to the naves ;
“ that is, to stand at right angles to the axles ;
“ because they would then bear the weight of the
“ load perpendicularly, which is the strongest way
“ for wood. But because the ground is generally
“ uneven, one wheel often falls into a cavity or rut
“ when the other does not ; and it bears much
“ more of the weight than the other does ; in
“ which case, concave or dishing wheels are best,
“ because when one falls into a rut, and the other
“ keeps upon high ground, the spokes become
“ perpendicular in the rut, and therefore have the
“ greatest strength when the obliquity of the load
“ throws most of its weight upon them ; whilst
“ those on the high ground have less weight to
“ bear, and therefore need not be at their full
“ strength. So that the usual way of making the
“ wheels concave is by much the best.”

“ The axles of the wheels ought to be perfectly
“ straight, that the rims of the wheels may be
“ parallel to each other ; for then they will move
“ easiest, because they will be at liberty to go on
“ straight forwards. But in the usual way of prac-
“ tice, the axles are bent downward at their ends ;
“ which brings the sides of the wheels next the
“ ground

“ ground nearer to one another, than their opposite or higher sides are : and this not only makes
“ the wheels drag sidewise as they go along, and
“ gives the load a much greater power of crushing
“ them than when they are parallel to each other,
“ but also endangers the overturning of the carriage when any wheel falls into a hole or rut, or
“ when the carriage goes in a road which has one
“ side lower than the other, as along the side of a
“ hill ;” * for on that construction the carriage stands upon a narrower base, than when the rims of the wheels are parallel to each other.

Upon level ground a carriage with four equal wheels may be drawn by the same power with equal facility, whether the load be placed on any particular part of the carriage, or it be spread equally all over it.

Upon a two wheeled carriage the most advantageous disposition of the load is, when the centre of gravity of the weight coincides with the middle of the axle, or with a perpendicular line which passes through that middle.

A carriage having the two hind wheels large, and the two fore wheels small, when going upon an horizontal plane, should have the principal part of the load laid towards its hind part ; but when going upon uneven roads, up and down hill, and when the load cannot be easily shifted, the best

* Ferguson's Lectures, lecture. iv.

way is to lay the load principally in the middle, or to spread it equally all over the carriage.

The common practice of carriers, who place the principal part of the load upon the fore axle of the waggon, is evidently very absurd; for by that means they press that axle with greater force upon the wheels, and the fore wheels deeper into the ground, in consequence of which those wheels, which turn oftener round than the large wheels, will not only wear much sooner, but require a greater power to draw them along, and especially over any obstacle.

The lower the centre of gravity of the load is situated, the less apt is the carriage to be overturned. The following observations are of Professor Anderson of Glasgow.

“ In Glasgow and its neighbourhood, a single horse, on a level turnpike road, draws 25 cwt. in a cart which weighs about 10 cwt. having wheels six feet high, and its axle passing through the centre of gravity of the load and cart, but, in a common cart, he draws only the half of that load. Two horses yoked in a line, in a common cart easily draw 30 cwt. upon an even road. And six horses, yoked two abreast, draw 80 cwt. in a common waggon.”

“ Six horses, in six carts, with high wheels, can draw 150 cwt. on a level road; and six horses, in three common carts, with two horses in each, can draw, upon an uneven road, 90 cwt. that

“ is 10 cwt. more than they can do in a waggon ;
 “ the weight, tear and wear, and the ease in draw-
 “ ing a waggon, or three carts, being, it is said,
 “ nearly equal ; and the price of the three carts
 “ being less than that of the waggon.” *

CHAPTER XVI.

OF PROJECTILES.

WHATEVER body is impelled by any power, and is afterwards left to proceed by itself, is called a *projectile*, which denomination is derived from a Latin word, the meaning of which is to throw, to hurl. Thus the bullets which are thrown out of fire arms, stones that are thrown by the hand, or by a sling, or by any other projecting instrument, &c. are called *projectiles*.

It has been already shewn (in chap. IX.) that projectiles, unless they be thrown perpendicularly upwards or downwards, must describe a curve line ; because they are acted upon by two forces, one of which, viz. the impelling force, produces an

* Institutes of Physics : Mech. sect. xvii.

equable motion ; whilst the other, viz. the attraction of the Earth, produces an accelerated motion.

It has likewise been shewn that projectiles describe such curves as are called by the mathematicians *parabolas* ; or rather that they would describe such curves, if they were not influenced by certain fluctuating circumstances, which cause the paths of projectiles to deviate more or less from true parabolic curves.

Thus much might have sufficed with respect to the motion of projectiles. . But the great use which is made of them both in peace and in war, obliges us to consider this branch of the doctrine of motion in a more particular manner, and to derive from the theory such rules as may be of use in the practical management of projectiles.

There are three causes, which force the projectile to deviate from the parabolic path : viz. 1st, the force of gravity's not acting in directions perpendicular to the horizon ; 2dly, The decrease of the force of gravity according to the squares of the distances from the centre of the Earth ; and 3dly, the resistance of the air.

The effects which are produced by the first and second of those causes, are too small and trifling ; for the centre of the Earth is at so great a distance from the surface, that both the height and the distance to which we are able to throw projectiles, are exceedingly small in proportion to it.

But the resistance of the air offers a very considerable opposition to the motion of projectiles, and its action is so very fluctuating, that, notwithstanding the endeavours of several able philosophers and mathematicians, the deviation of projectiles from the parabolic path has not yet been subjected to any determined and practical rules.

After the preceding remarks it will be easily allowed, that the only method we can follow is to lay down the theory on the supposition that the projectiles move in parabolas, and then to subjoin a concise statement of the result of the principal experiments, which have been made for the purpose of shewing the deviation of the real path of a projectile from the parabolic curve.

When a body is projected obliquely from any kind of engine, such as a ball from the cannon A, fig. 3, Plate IX. in the direction AC, the force of gravity, acting upon it in directions nearly perpendicular to the horizontal plane AB, forces it to deviate from the straight direction AC, and to describe the parabola ADB, lastly falling upon the horizontal plane at B; whence it evidently follows, that a ball or any other projectile cannot move even for a moment in a straight line; but that it must deviate more or less from the straight line of its initial direction, and must immediately begin to incline towards the ground; excepting however some particular cases; namely, when the shot has acquired a rotatory motion

tion round its axis, or when its shape is somewhat oblong and bent; for in those cases it may deviate not only sideways, but even upwards for a short time.

The distance AB between the mouth of the projecting engine and the place where the shot falls upon the horizontal plane, is called the *range of the shot*, or the *amplitude of projection*; DE is the *height* of its path, or of the parabola ADB. The angle CAB, which the direction of the projection, or of the cannon, makes with the horizontal plane, is called the *angle of elevation*. The time during which the shot performs the path ADB, is called *the time of flight*, and the force with which it strikes an object at B, is its *momentum*.

It will be found demonstrated in the note that those particulars, viz. the range, the height, the angle of elevation, &c. bear a certain determinate proportion to each other, so that when two of them are known, the others may thereby be found out. It is demonstrated likewise, that, *cæteris paribus*, the greatest range or greatest distance to which a shot may be thrown upon an horizontal plane, takes place when the angle of elevation is equal to half a right angle, or 45 degrees (1.); we shall

(1.) Proposition I. *A body which is projected in a direction not perpendicular, but oblique, to the horizon, will describe a parabola; and its velocity in any point of that parabola*

shall therefore proceed in this place to shew a practical method of determining the most useful of those particulars, when the angle of elevation, and the greatest range the cannon is capable of, are known.—This greatest range, viz. the distance to which the cannon when charged with the usual quantity of powder, and elevated to an angle of 45° . is capable of throwing the shot, must be ascertained by means of actual experiment and mensuration in every piece of artillery, especially with large cannons, and mortars; since the balls from those pieces deflect much more from the straight direction.—The angle of elevation is ascertained by means of a graduated circular instrument, and a plummet, or a level.

Let

la is the same as it would acquire by descending perpendicularly through a space equal to the fourth part of the parameter belonging to that point as a vertex; supposing that the force of gravity is uniform, and that it acts in directions perpendicular to the horizontal plane; also that the air offers no resistance to the motion of projectiles.

Let a body be projected from A, fig. 5. Plate IX. in the direction AE, and let AE represent the space, through which the projecting force alone would carry it with an equable motion in the time T. Also let AB represent the space through which the force of gravity alone would cause it to descend in the same time T. Complete the parallelogram ABEC, and it is evident that the body, being impelled both by the projecting force, and by the force of gravity,

Let the greatest horizontal range of a cannon, or mortar, be 6750 yards, and let the actual angle of elevation be 25° ; the other particulars may be found by delineating this case upon paper; for which purpose the instruments that are generally put

gravity, must, at the end of the time T , be found at C . Now AE is as the time T , because it represents the space described uniformly; but AB is as the square of the time T ; therefore AE or its equal BC , is as the square of AB . And the same reasoning may be applied to any other contemporary distances, as AH , AF , or FG , AF . But AE is a tangent to the curve at the point A , AF is a diameter at the point A , and BC , FG , &c. being parallel to the tangent AE , are ordinates to the diameter AF ; and since the squares of those ordinates have been demonstrated to be as the respective abscissas AB , AF , &c.; therefore the curve $ACGD$ is the parabola.

The velocity of the projectile at any point, as A , in the curve is such, that the space AE would be described uniformly by it, in the same time that the body would employ in descending perpendicularly by the force of gravity from A to B . Also (see p. 66.) the velocity acquired by the perpendicular descent AB is such as would carry the body equably through twice AB in the same time, (that is, in the same time that AE is described;) therefore the velocity which is acquired by the perpendicular descent AB , is to the velocity with which AE is described, as twice AB is to AE . But the velocity acquired by the perpendicular descent through AB , is to the velocity acquired by the perpendicular descent through a quarter of the parameter belonging to the vertex A of the parabola ACG , also as

put in a common case of drawing instruments, are quite sufficient, viz. a pair of compasses, a ruler with a scale of equal parts, and a protractor.

Draw an indefinite right line AK, fig. 4. Plate IX. to represent an horizontal plane, passing through

twice AB is to AE; for that parameter is (by conics) equal to $\frac{AE^2}{AB}$, and a quarter of it is $\frac{AE^2}{4 AB}$, and, by the

laws of gravity, the velocity acquired by the perpendicular descent AB, is to the velocity acquired by the perpendicular descent $\frac{AE^2}{4 AB}$, as the square roots of those spaces; viz.

as $AB^{\frac{1}{2}}$ is to $\frac{AE}{2 BA^{\frac{1}{2}}}$; or as twice AB is to AE. There-

fore, since the like reasoning may be applied to any other point of the parabolic path, we conclude that, universally, the velocity of the projectile in any point of its path is the same as would be acquired by a perpendicular descent through a space equal to the fourth part of the parameter belonging to that point as a vertex.

Corollary 1. It is evident that the projectile must move in the plane of the two forces, viz. in the plane which passes through AE, AB, and is, of course, perpendicular to the horizon.

Cor. 2. It follows from the laws of compound motion, that the projectile will describe the arch AC, in the same time in which it would descend by the force of gravity from A to B, or in which it would describe uniformly the space AE.

Cor. 3. When a body is projected from A in the direction AE, if the parameter which belongs to the vertex A, be

through the point of projection A. Make AB perpendicular to it, and equal to twice the greatest horizontal range, viz. equal to 13500 yards; which is done by making it equal to 13500 divisions of the

be equal to $\frac{AE^2}{EC}$, the parabola must pass through the point C.

Cor. 4. Either in the same, or in different parabolas, the parameters belonging to different points are to each other as the squares of the velocities of the projectile at those points (see p. 65); whence it follows, that at the vertex of the parabola the velocity or the momentum of the projectile is the least, and at equal distances from that vertex the velocities or the momentums are equal.

Proposition II. *The initial velocity being given, to find the direction in which a body must be projected in order to hit a given point.*

Let A, fig. 6, Plate IX, be the projecting point, and C the object, or point which is required to be hit.

The velocity of projection being given, the parameter of the parabola which must pass through the point C will easily be found by means of the preceding proposition; viz. by finding the space, through which a body must fall from rest, in order to acquire the given velocity; for that space is equal to the fourth part of the parameter belonging to the point A.

Join AC, draw the horizontal line AL, and at A erect AP perpendicular to the horizontal line AL, and equal to the above-mentioned parameter. Divide AP into two equal parts at G, and through G draw an indefinite right line KGH parallel to the horizontal line AL. Through A draw

the scale of equal parts, for those parts must represent yards. Upon AB , as a diameter, describe the semicircle AFB . At A , by means of the protractor, draw the line of projection AF , making an angle of 25° with the horizontal line AK .

Through

draw AK perpendicular to the direction AC of the object, which AK will meet KH in a point K . With the centre K and radius KA draw the circular arch $PHEA$. Through the point or object C draw BCI perpendicular to the horizon, and if this perpendicular meets the circular arch, as at E and I , draw AE , AI ; and either of those directions will answer the desired purpose.

Join HI and PE ; and the triangles PAE , EAC are similar; the angle PAE being equal to the angle AEC (Eucl. p. 29, B. I.) and the Angle APE equal to the angle EAC (Eucl. p. 32, B. III.) Hence $PA : AE :: AE : EC$;

therefore $PA = \frac{AE^2}{EC}$. Farther, the triangles PAI ,

AIC are also similar; the angle PAI being equal to the angle AIC (Eucl. prop. 29, B. I.) and the angle API equal to IAC (Eucl. p. 32, B. III.) Hence $PA : AI :: AI :$

IC ; and $PA = \frac{AI^2}{IC}$. Therefore since PA is the pa-

rameter belonging to the point A of the parabola, which is to be described by the projectile, &c. the said parabola (by cor. 3 of the preceding prop.) must pass through the point C .

Corollary 1. The angular distance CAP between the object and the zenith, is divided into two equal parts by the line AH ; for KH being equal to KA , the angle AHK is equal to the angle HAK , and likewise equal (on account

of

Through the point F , where AF cuts the semicircle, draw OF perpendicular to the horizon AK . Divide AO into two equal parts, and at the point of division C erect CD perpendicular to the horizontal line AK ; also make CD equal to a quarter

of the parallelism of KH , AB) to the Angle HAB . But KAC is equal to GAB ; for they are both right angles; therefore, subtracting the angle GAC from both, there remains KAG equal to CAB ; consequently GAH is equal to HAC .

Cor. 2. The two directions AI , AE are equidistant from the direction AH ; for KH being perpendicular to PA and to IE , the arch AH is equal to HP . and EH is equal to HI .

Cor. 3. When the directions AI , AE coincide with AH , then the distance AC is the greatest distance to which the projectile can be thrown upon the plane AO with the given velocity of projection. Hence it appears that when the object C is placed upon the horizontal plane, as in fig. 7, Plate IX. where AC coincides with AB , AK coincides with AG , PHA becomes a semicircle, and HAC half a right angle; then the greatest distance to which the projectile can be thrown, takes place; viz. when the angle of elevation HAB is half a right angle.

Cor. 4. The velocity of projection being known, the greatest distance AL , to which the projectile can be thrown upon the horizontal plane, or greatest range, is likewise known; it being equal to half the parameter AP ; for AL is equal to the radius KH , or AK , which is the half of AP .

Cor. 5. When the point of projection A , and object C , are both upon the same horizontal plane, as in fig. 7; then

ter of OF; then the path of the shot is represented by a curve line, which passes through the points A, D, O. Take the distance AO in your compasses, and, applying it to the same scale of equal parts as was used before, you will find it equal to 5170, which represent yards. If you apply the distance
CD

then the distance AC of the object is as the sine of twice the angle of elevation CAE, or CAI; for (Eucl. p. 32, B. III.) CAE is equal to APE, and likewise equal (Eucl. p. 20, B. III.) to half AKE, whose sine is EN; and EN is equal to AC. EN is likewise the sine of double the angle CAI; for $CAI = API = \frac{1}{2} AKI$; and $IS = NE$, is the sine of the angle AKI.

Cor. 6. If AE be the direction of the projectile, the greatest height of the parabolic path above the horizon, is equal to a quarter of AC, and is as the versed sine of twice the angle of elevation CAE. For divide AC into two equal parts at T, and erect TR perpendicular to it. Divide TR into two equal parts at V; then TV is equal to half TR, and to a quarter of EC. It is evident that AC is an ordinate to the axis TR of the parabolic path; and that V must be the vertex of that parabola; for the direction AE being a tangent to the parabolic curve, the part VT of the axis is (by conics) equal to the part VR. Farther, CE is equal to AN, which is the versed sine of the angle AKE, viz. of twice the angle of elevation EAC, or IAC; therefore TV is equal to a quarter of EC, and is as the versed sine of twice the angle of elevation CAE.

Cor. 7. The greatest height, to which the projectile will ascend, when the direction of the projection is perpendicular

CD to the same scale, it will be found equal to 603 divisions, or yards; which shews that with the angle of elevation equal to 25° , the cannon in question will throw the shot to the horizontal distance of 5170 yards, and that the vertex, or greatest

dicular to the horizon, is equal to a quarter of the parameter AP; for in that case AE, EC, and AP coincide; consequently a quarter of CE becomes the same thing as a quarter of AP.

Cor. 8. In the case of the preceding corollary, the time of the projectile's remaining in the air, is the same that a body would employ in descending from P to A, merely by the force of gravity; for the projectile will be as long in ascending, as in descending along a quarter of AP; viz. it will employ twice the time which is required to descend along one quarter of PA, and which is equal to the time that is required to descend from P to A; the spaces described by descending bodies being as the squares of the times.

Cor. 9. The time of flight when a body is projected in any direction, as for instance AE, is as the sine of the angle of elevation EAC; for it is as the chord AE, or as half AE, which is the sine of the angle APE, or of half AKE, which is equal to the angle of elevation CAE.

From the abovementioned two propositions, with their corollaries, the most useful properties of projectiles have been derived, and are concisely expressed together with the results of experiments, in the following practical rules, which every gunner should impress in his mind; for when the greatest range that a piece of artillery is capable of with the usual charge of powder, is known, and which must be

learned

est height of its path ADO will be equal to 603 yards.

By the like means the range answering to any other angle of elevation may be ascertained; and
the

learned from actual experiment; those rules will answer all the necessary cases in gunnery; excepting the obstruction which arises from the resistance of the air.

Rules for Shooting.

1. Horizontal ranges as right sines of twice the angles of elevation.
2. Heights as versed sines of twice the angles of elevation.
3. Times of flight, or times in the air, as right sines of the angles of elevation.
4. The time of flight at an elevation of 45° , is equal to the time of perpendicular descent through a space equal to the horizontal range.
5. The impetus is equal half the horizontal range at 45° of elevation.
6. The height is equal to a quarter of the horizontal range at 45° of elevation.
7. In ascents or descents, (viz. when the point of projection and the object are not both upon the same horizontal plane) for the best elevation take the complement of half the angular distance between the object and the zenith.
3. The charges of powder in the same piece are nearly as the horizontal ranges.

The instruments which are required for the practical application of those rules, are a graduated circular instrument with a plummet or a level, and a table of sines and versed sines;

the reason of this practice will be found demonstrated in the note.

If, on the other hand, the distance of the object, and the cannon's greatest range being known, the angle of elevation necessary to hit that object be required; you must proceed by the reverse of the preceding method. Let, for instance, the distance of the object from the cannon be 5170 yards, and the cannon's greatest range, 6750 yards. Draw a right line AO, fig. 4. Plate IX. equal to 5170 divisions of the scale of equal parts. Make AB perpendicular to AO, and equal to twice the cannon's greatest range; viz. to 13500 divisions of the same scale. Upon AB, as a diameter, describe the semicircle AFB. At O erect the line OG perpendicular to the horizontal line AO, and the perpendicular OG will meet the semicircle either in one, or in two points, or not at all. Should this line meet the semicircle in one point, it must be at Y, its middle, and then the required angle of elevation is YAO, viz. of 45° . If OG

fines; but there is an instrument in use, called *the gunner's callipers*, which answers every purpose relative to the application of those rules; as it contains a graduated circle, and is susceptible of the application of a plummet, &c. A table of fines and versed fines, together with many other tables and measures are likewise marked upon it.—See a very good description, and account of the various uses of the gunner's callipers, in Robertson's *Treatise on the Use of Mathem. Instrum.*

meets

meets the semicircle in two points, as at F and G, which is the case in the present instance; then either the direction AF or AG will answer the purpose; and if the angles which those directions make with the horizon AO, be measured by means of the protractor, the former will be found equal to 25° , and the latter to 65° . But when the perpendicular OG does not meet the semicircle, then we must conclude that the given power, or force of the cannon in question, is not sufficient to throw the shot to the proposed distance.

It follows from the foregoing construction, that the higher the direction line cuts the semicircle, the longer is the horizontal range; not exceeding however the middle point Y; for SY, which is equal to the radius of the circle, is longer than any other line that can be drawn in the semicircle AYB, parallel to the horizon; hence the horizontal range is the greatest when the angle of elevation is 45° . It is also evident, that at equal distances from the point Y, as at F and G, the horizontal range will be the same; the height of the path only being different. When the angle of elevation is 90° , then the line of direction, becoming perpendicular to the horizon, coincides with the line AB, and the horizontal range becomes equal to nothing, viz. the shot will ascend to a quarter of AB, and will then descend again to A.

When the angle of elevation is half a right angle, that is 45° . the time of flight, viz. of the

shot's remaining in the air, is equal to the time that a body would employ in descending perpendicularly, by the force of gravity, from a height equal to the horizontal range, which may be found by the rules given in chap. V.—In order to find the time of flight, at any other inclination, as for instance, the inclination OAF; say as AY (measured on the scale of equal parts) is to AI (also measured on the same scale), so is the time that a body would employ in descending perpendicularly through a space equal to the greatest horizontal range, to the time in question, which will be known by the common rule of three.

Thus much may suffice with respect to the supposed parabolic paths of projectiles. We shall now subjoin a short account of the result of such experiments as have been made for the purpose of determining how far the parabolic theory may be depended upon.

In the common practice of directing cannons, an arbitrary allowance is made for the deviation of the shot from the straight line. Practice indeed renders some gunners very expert; but their practical accuracy cannot be reduced to certain rules; viz. such as may be of use to other persons.

Mortars, which throw the shots in general with less velocity than cannons, have heretofore been directed by means of a graduated instrument; preferring, of the two directions which produce the same horizontal range, that which may be thought preferable according to the nature of the object.

But of late another method has been found more advantageous in practice.—The mortar is steadily fixed upon its bed or carriage, at an elevation of 45° ; but it is loaded with more or less powder according as the shell is required to go farther or nearer; not exceeding, however, the greatest horizontal range the piece is capable of; the errors of amplitude having been found to be less with an elevation of 45° , than with any other elevation; and the horizontal ranges having been found to be, *cæteris paribus*, nearly as the charges of powder; viz. half the weight of the full charge will throw the shell nearly to half the greatest horizontal range; a quarter of the weight of the full charge will throw the shell to a quarter of the greatest range, &c.

It has been found that a 24 pounder (viz. a cannon whose ball weighs 24 pounds) when charged with 16 pounds of gunpowder, and elevated to an angle of 45° , will generally range its shot upon an horizontal plane 20250 feet, which is not above one fifth of the range assigned by the theory, viz. of what it ought to be, if the air could be removed.—The opposition which the shot meets with from the air in this case has been estimated by the ingenious Mr. Robins, equivalent to 400 pounds.* Hence it appears that the path of a shot is far different from the parabola.

* Robins's Essays on Gunnery.

The resistance of the air varies principally according to its fluctuating qualities, viz. temperature, gravity, &c. according to the shape of the shot, and according to the velocity with which the shot is impelled, viz. the initial or incipient velocity.—When that velocity is small, the resistance of the air is very trifling; but when the initial velocity is very considerable, the resistance of the air becomes so great as to render the theory quite inapplicable to practice. The resistance sometimes amounts to 20 or 30 times the weight of the shot, and the horizontal range frequently is much less than the tenth part of what it ought to be, according to the parabolic theory.

It has been found that with the same angle of elevation, the horizontal ranges are in proportion to one another as the square roots of the initial velocities, and that the times of flight are as the ranges; whereas, according to the theory, the times ought to be as the velocities, and the ranges as the squares of the initial velocities.

Mr. Robins likewise found that very little advantage was gained by projecting a body with a velocity greater than 1200 feet per second. When a 24 pound shot is projected with the velocity of 2000 feet in a second, it will meet with so great an opposition from the air, that when it has advanced not more than 1500 feet, viz. in about one second, its velocity will be reduced to that of about

1200 feet per second. In consequence of this quick reduction of velocity, Mr. R. concluded that a certain projectile velocity at the same angle, might carry a shot farther than a greater velocity; for the body projected with the greater velocity, when its velocity becomes equal to that of the other projection, has a less angle of elevation, on which account it may go not so far from that point, so as to make the whole distance shorter.

No gun to carry far should be charged with powder whose weight exceeds one sixth, or at most one fifth part of the weight of the shot; for in field pieces that quantity of powder will impel the shot with the initial velocity of about 1200 feet per second. In a battering piece, when the object is near, the weight of the powder should be about one third part of the weight of the shot.

When the initial velocity is greater than about 1100, or 1200 feet per second, the resistance of the air seems to be three times greater than it ought, if it varied only as the square of the velocity. "The velocity at which the variation of the law of resistance takes place, is nearly the same as that with which sound moves. Indeed if the treble resistance in the greater velocities is owing to a vacuum being left behind the resisted body, it is not unreasonable to suppose that the celerity of sound is the very last degree of celerity with which a projectile can form this vacuum, and

“ can in some sort avoid the pressure of the atmosphere on its hinder parts. It may perhaps confirm this conjecture to observe, that if a bullet, moving with the velocity of sound, does really leave a vacuum behind it, the pressure of the atmosphere on its fore part is a force about three times as great as its resistance, computed by the laws observed for slow motions.*”

A shot, besides its being drawn downwards from the line of direction, is sometimes deflected sideways; which takes place when the shot by rubbing against one side of the cavity of the piece, acquires a rotatory motion round its axis, and proceeds through the air with that motion; for in that case the side of the shot, which in its course through the air turns forwards, meets with greater resistance than the opposite side, whose motion coincides with that of the air. It is easy to conceive that when the axis of rotation happens to be parallel to the horizon, then the rotation will contribute to the shot's deflection, not sideways, but upwards or downwards.

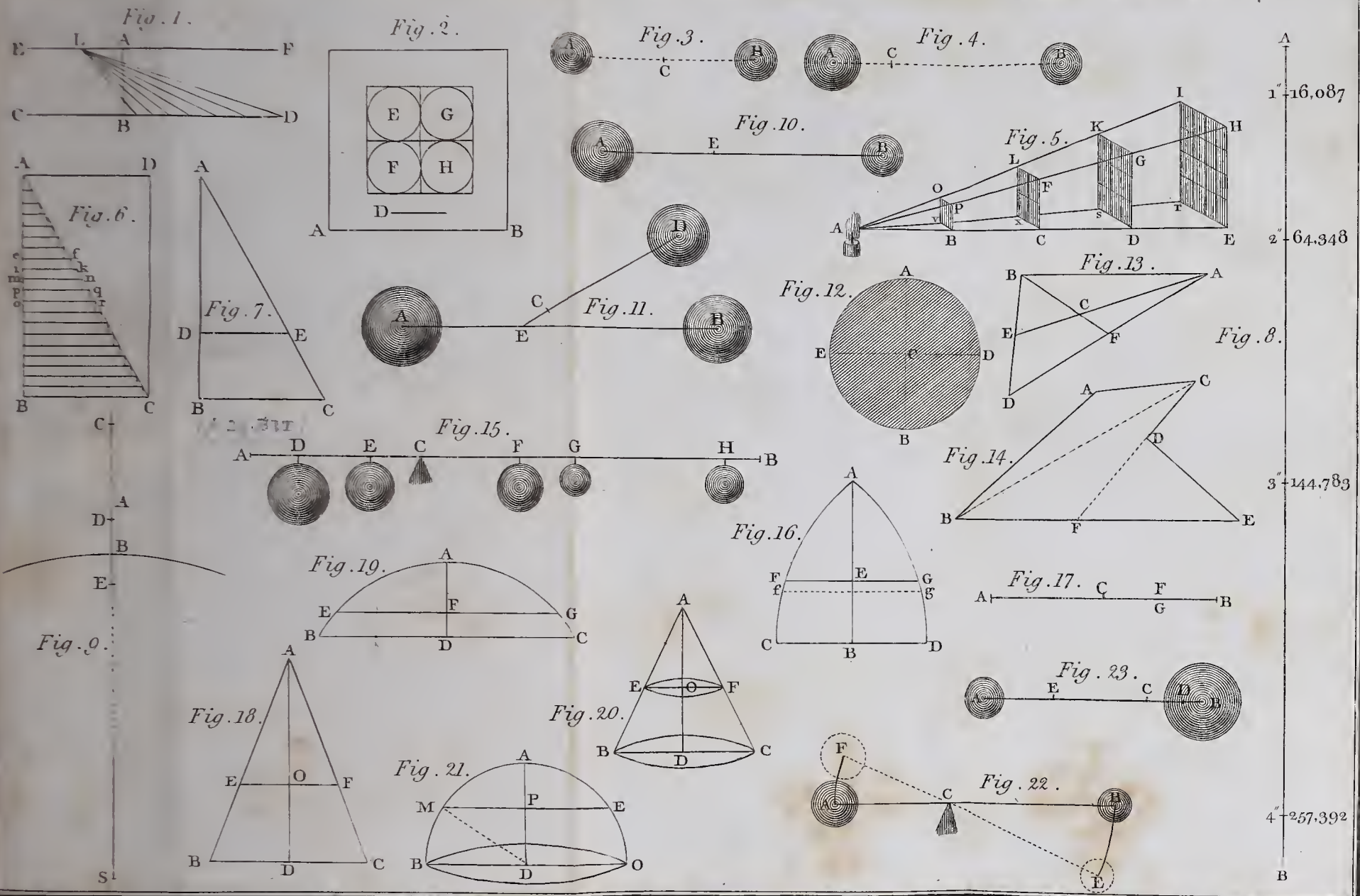
We shall lastly observe as a strong instance of

* Robins's Essays on Gunnery.—The reader may derive considerable information from Dr. Hutton's Paper on the Force of fired Gunpowder, and the initial Velocities of Cannon Ball, &c. in the 68th vol. of the Phil. Trans. And from the Chev. de Borde's Memoirs in the Hist. of the Acad. of Scienc. for 1769.

the difficulty which attends the practical management of projectiles, that even with the very same piece of ordnance, like shots, equal weights of the same sort of gunpowder, and the same angle of elevation, the places on which the shots strike the horizontal plane frequently differ by several yards from each other.

END OF THE FIRST VOLUME.

Luke Hansard, Printer,
Great Turnstile, Lincoln's-Inn Fields.



1" 16.087
2" 64.348
3" 144.783
4" 257.392



Fig. 2.

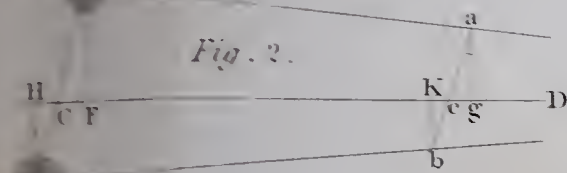


Fig. 3.

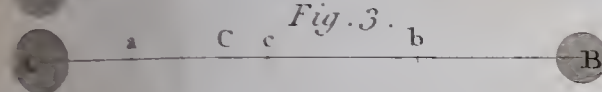


Fig. 4.

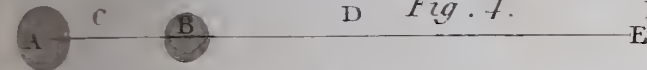


Fig. 5.



Fig. 6.

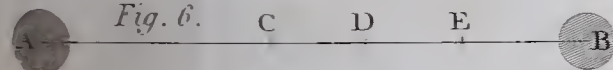


Fig. 7.



Fig. 8.

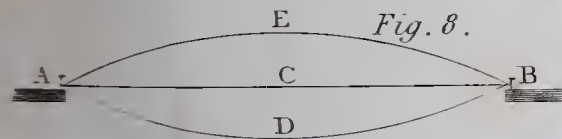


Fig. 9.

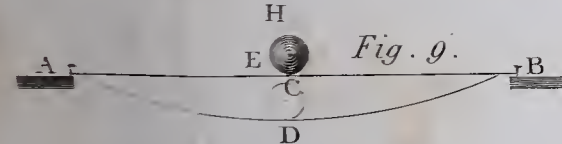


Fig. 10.

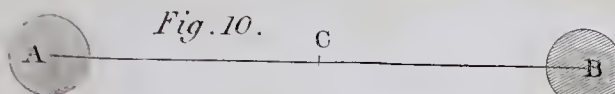


Fig. 11.

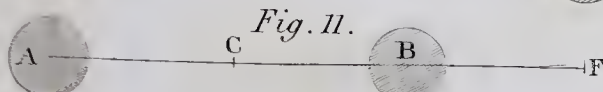


Fig. 12.

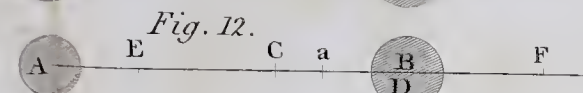


Fig. 13.

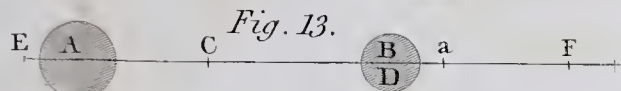


Fig. 14.

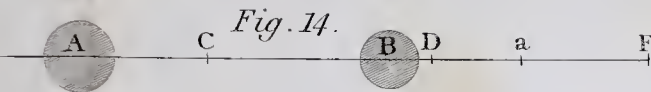


Fig. 15.

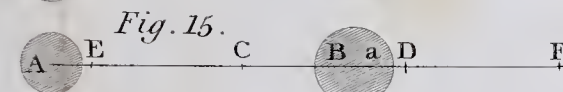


Fig. 16.



Fig. 17.

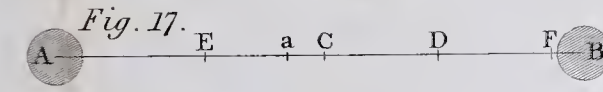


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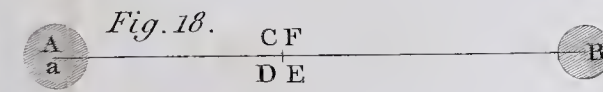


Fig. 19.



Fig. 20.

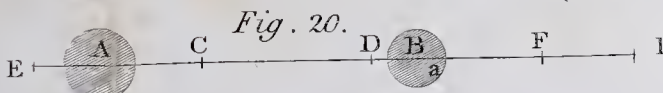


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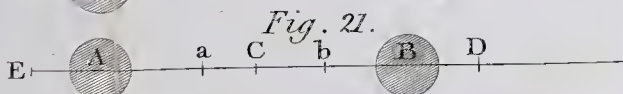


Fig. 22.

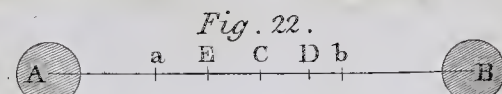


Fig. 23.



Fig. 24.

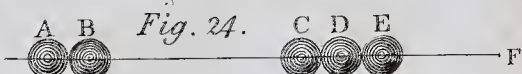


Fig. 25.

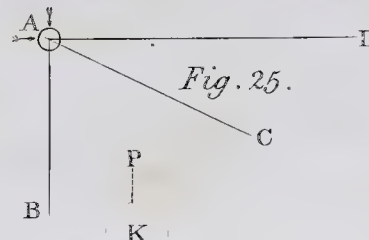


Fig. 26.

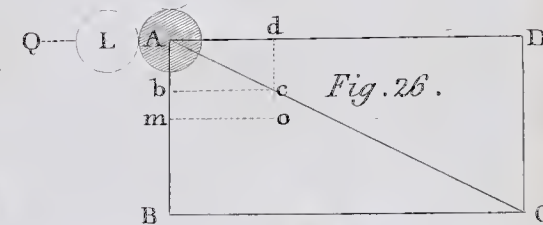


Fig. 27.

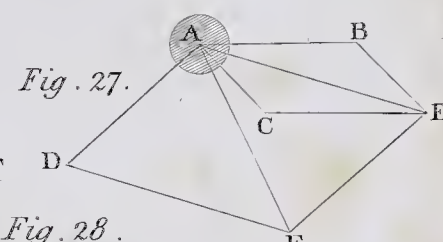


Fig. 28.

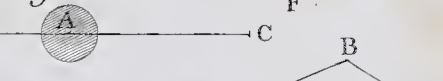


Fig. 29.



